Sources and Studies in the History of Mathematics and Physical Sciences

K. Ramasubramanian
Takao Hayashi
Clemency Montelle *Editors*

Bhāskara-prabhā

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Sources and Studies in the History of Mathematics and Physical Sciences

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Transliteration scheme

The transliteration scheme followed in this book to represent *Devanāgarī* script is given below.

Table of vowels

अ	आ	इ	ई	उ	ऊ	ऋ	ॠ	ऌ	ए	ओ	ऐ	औ	अं	अः
a	*ā*	*i*	*ī*	*u*	*ū*	*ṛ*	*ṝ*	*ḻ*	*e*	*o*	*ai*	*au*	*aṃ*	*aḥ*

Table of consonants

क *ka*	ख *kha*	ग *ga*	घ *gha*	ङ *ṅa*	च *ca*	छ *cha*	ज *ja*	झ *jha*	ञ *ña*
ट *ṭa*	ठ *ṭha*	ड *ḍa*	ढ *ḍha*	ण *ṇa*	त *ta*	थ *tha*	द *da*	ध *dha*	न *na*
प *pa*	फ *pha*	ब *ba*	भ *bha*	म *ma*					
य *ya*	र *ra*	ल *la*	व *va*	श *śa*	ष *ṣa*	स *sa*	ह *ha*	ळ *ḷa*	

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Preface

विबुधाभिवन्दितपदो जयति श्रीभास्कराचार्यः

Victorious is the illustrious Bhāskarācārya
whose feet/words are revered by scholars/gods

Thus proclaims an inscription set up by Bhāskarācārya's grandson Caṅgadeva. Indeed, from the middle of the twelfth century onwards, Bhāskara's magisterial writings have been revered through all the centuries in all parts of India, and he is respectfully referred to as Bhāskarācārya.

Therefore, when I visited Prof. S. R. Sarma in Düsseldorf, Germany, in April 2013, we decided to celebrate the 900th birth anniversary of Bhāskarācārya in 2014 at the Vidya Prasarak Mandal (VPM), Thane. On my return to India I consulted the trustees and members of the management committee of the VPM, and they readily agreed to the idea.

Prof. Sarma immediately started planning the details of the academic part of the proposed international conference. An advisory committee was formed with Prof. Michio Yano (Kyoto Sangyo University, Kyoto), Prof. Takao Hayashi (Doshisha University, Kyoto), Prof. K. Ramasubramanian (Indian Institute of Technology, Bombay), Dr. Arvind P. Jamkhedkar (Retired Director, Department of Archeology and Museums, State of Maharashtra) and Prof. Sudhakar Agarkar (Dean, VPM's Academy of International Education and Research), along with Prof. Sarma and myself. An organizing committee was also established which included the Principals and Directors of different institutions within the Vidya Prasarak Mandal. The Institute of Oriental Studies, Thane, also agreed to join as a co-organizer.

In addition to an international conference, it was decided to celebrate Bhāskarācārya's birth anniversary with outreach activities throughout the year. This outreach was to consist of workshops on the works of Bhāskarācārya,

in particular, on the *Līlavatī*, in schools and colleges run by the VPM and other organizations. We deemed this to be very important because our educational curriculum pays hardly any attention to the scientific heritage of the nation. This plan of addressing the students of schools and colleges through the workshops and the international community of science-historians through the conference was very much in the spirit of what Bhāskarācārya wished to achieve through his writings. In each of his texts, he repeatedly states that his works are meant to be accessible to beginners (*bāla-līlāvagamya*) but commanding of respect and appreciation from experts (*vidagdha-gaṇaka-prīti-prada*).

Prof. Sudhakar Agarkar took upon himself the onerous task of conducting almost sixty-five workshops single-handedly during the calendar year 2014 in various parts of Maharashtra, Madhya Pradesh and Andhra Pradesh to diverse kinds of groups. The participants included school children, college students, teachers and bankers. There has been a wonderful response to the workshops and Prof. Agarkar deserves hearty appreciation from the academic community for this laudable work.

The international conference was held during 19–21 September 2014 with forty registered participants both from India and abroad, which included countries like Canada, France, Japan, New Zealand, and the United States. The conference was inaugurated by the well-known nuclear scientist Prof. Anil Kakhodkar and the valedictory address was delivered by the eminent computer scientist Prof. Deepak Phatak of the Indian Institute of Technology, Bombay.

It is customary in India to commence conferences of this nature with a musical rendering of beautiful Sanskrit verses as an invocation. Since Bhāskarācārya's works themselves contain several beautiful invocatory verses, it was decided to commence every major session including plenary lectures with a recital of these verses by scholars and professional musicians. At the end of the academic sessions, there were cultural programmes every day. On two evenings the students of Vidya Prasarak Mandal's Joshi Bedekar College, Thane, presented dance and music programmes. On one evening there was a scintillating dance drama entitled the "*Lilavati Ganitham*" specifically tailored and choreographed for the conference. This was performed by the team from the Rishi Valley School, Madanapalli, Andhra Pradesh. The conference was held at the environment-friendly sprawling campus of the Thane College of the Vidya Prasarak Mandal. I am thankful to the Principals, Directors and all the student volunteers of the college who worked hard to make the conference a success.

The lectures of the conference are available in audio-visual format at the Bhaskara 900 website (http://www.vpmthane.org/bhaskara900/). Electronic

copies of Bhāskara's works and important studies related to them are also available at this website.

A selection of the papers presented at the conference are collected in this volume of proceedings, appropriately named *Bhāskara-prabhā* (The Radiance of Bhāskara). It gives me great pleasure to present this volume to the academic community, on behalf of the editorial team and myself. My colleagues on the editorial board, Professors Takao Hayashi, Clemency Montelle and K. Ramasubramanian have put in great efforts in meticulously editing this volume. To them and to the supporting team at the IIT Bombay our warmest thanks. Thanks are also due to Prof. Sudhakar Agarkar, Dr Arvind Jamkhedkar, Prof. S. R. Sarma and Prof. Michio Yano for their invaluable help. Last but not least, our sincere thanks and appreciation go to all the participants of the conference.

June 24, 2018
Mumbai

Vijay Bedekar
Chairman
Vidya Prasarak Mandal, Thane

Acknowledgements

Editing a collection of scholarly contributions is a large and complex job. We wish to express our sincere gratitude to Prof. S. R. Sarma for his vision, his overall careful and meticulous planning of the various topics according to which the contributions were arranged, and his direction throughout the whole process. We also wish to thank him for his efforts to see that the Caṅgadeva inscription along with background historical details were brought to light and included in the volume.

In the process of finalizing the text for publication, the manuscripts went through several rounds of revisions and the corrections that had to be carefully implemented. We would like to express our sincere appreciation to several people involved in this process who tirelessly assisted us in this regard. First and foremost, Sushama Sonak for her overall coordination of the entire team and meticulously going through many rounds of proof corrections, to Sreelekshmy Ranjit, Lalitha Hotkar, G. Periyasamy, and Dr Dinesh Mohan Joshi for typesetting and the implementation of corrections, and to Dr K. Mahesh, Aditya Kolachana and Devaraja Adiga for their technical assistance.

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कल्यब्द ५११९

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In *Kaṭapayādi*: शौरिदीधितिर्हृद्या

Thursday
June 21, 2018

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Introduction

Among the galaxy of mathematicians who flourished on the Indian subcontinent, Bhāskarācārya holds a very special place. Born in 1114 CE, Bhāskara wrote a collection of works on mathematics and astronomy which not only contained many technically brilliant results, but delighted readers with their poetic sophistication, imaginative appeal, and enduring charm. From the simple addition of two numbers, to elaborate algorithms accounting for subtleties in planetary motion, Bhāskara enchanted his audience as much with his lucidity and eloquence, as his singular insight, intellectual ingenuity, and skills as a gifted teacher.

Bhāskara alludes to these qualities in the opening verses of his monumental treatise on astronomy, the *Siddhāntaśiromaṇi*, arguably one of the most comprehensive and erudite astronomical treatises, where he declares (*Siddhāntaśiromaṇi*, ch. 1.3):

कृत्वा चेतसि भक्तितो निजगुरोः पादारविन्दं ततः
लब्ध्वा बोधलवं करोति सुमतिप्रज्ञासमुल्लासकम् ।
सद्वृत्तं ललितोक्तियुक्तममलं लीलावबोधं स्फुटं
सत्सिद्धान्तशिरोमणिं सुगणकप्रीत्यै कृती भास्करः ॥

kṛtvā cetasi bhaktito nijaguroḥ pādāravindaṃ tataḥ
labdhvā bodhalavaṃ karoti sumatiprajñāsamullāsakam |
sadvṛttaṃ lalitoktiyuktamamalaṃ līlāvabodhaṃ sphuṭaṃ
satsiddhāntaśiromaṇiṃ sugaṇakaprītyai kṛtī bhāskaraḥ ||

The blessed Bhāskara composes the *Siddhāntaśiromaṇi*, [filled] with enchanting meters (*sadvṛttaṃ*), whose style of writing is simple (*lalitoktiyuktam*), besides being flawless and clear, aiming to explain concepts with playful elegance (*līlāvabodha*) to make the intellect of the smart ones fully blossom, having devoutly placed the lotus feet of his teacher into his heart, having gained merely an iota of knowledge from him, for the purpose of pleasing expert mathematicians (*sugaṇakaprītyai*).

With these words Bhāskara acknowledged his multiple desiderata above and beyond the articulation of astronomical rules and procedures: a composition to be filled with charming verses, easy to understand, flawless, clear, and written such that dedicated readers may advance their expertise with a sense of great joy in the learning process. Indeed, there is not a single scholar since who would doubt the importance and significance of this remarkable work, and pay due homage to Bhāskara's ability to cast even the most thorny astronomical concepts in an appealing and poetically remarkable way. Throughout the ages, commentators on his works have equally admired his literary flair as they have the brilliance and ingenuity of his rules. As early as the thirteenth century, we find testament to Bhāskara's mastery, in an inscription authored by Caṅgadeva (see Part I). Having extolled Bhāskara's intellectual accomplishments in various branches of traditional knowledge systems, Caṅgadeva concludes with this glorious tribute:

विबुधाभिवन्दितपदो जयति श्रीभास्कराचार्यः ।

vibudhābhivanditapado jayati śrībhāskarācāryaḥ |

Triumphant is the illustrious Bhāskarācārya whose feet/words (*pada*)[1] are venerated by the wise gods (*vibudha*).

The flow of time hasn't diminished the glory of Bhāskara's works. Admiration has continued over the centuries to recent times. For instance, D. Arkasomayaji, an eminent scholar of the twentieth century with expertise in both traditional *śāstras* and modern mathematics, in his dedication to the explanatory notes on the *Gaṇitādhyāya* of the *Siddhāntaśiromaṇi*, brings out the distinction of Bhāskara's work with the following beautiful imagery [SiŚi1980, pp. viii–ix]:

यः सिद्धान्तशिरोमणिं न पठति[2] श्रीभास्करार्यस्य यः
चात्मानं लघुपुस्तकेक्षणपरः चेत् पण्डितं मन्यते ।
रङ्गत्तुङ्गतरङ्गितं हिमगिरिप्रस्यन्दिगङ्गाजलं
हित्वा तेन सुपङ्किलं किल जलं स्नातुं समाश्रीयते ॥

yaḥ siddhāntaśiromaṇiṃ na paṭhati śrībhāskarāryasya yaḥ
cātmānaṃ laghupustakekṣaṇaparaḥ cet paṇḍitaṃ manyate |
raṅgattuṅgataraṅgitaṃ himagiriprasyandigaṅgājalaṃ
hitvā tena supaṅkilaṃ kila jalaṃ snātuṃ samāśrīyate ||

He, who does not study the *Siddhāntaśiromaṇi* of Bhāskarācārya, and feels that he is a scholar reading a few other substandard books on Hindu astronomy, does verily go to bathe in a dirty pond, ignoring the holy Gaṅgā, jumping down the heights of the Himalayas in surging and dancing billows!

[1] It may be noted here that the words *pada* and *vibudha* have been employed with a double entendre.

[2] Since the reading पठित: in the published edition of the text is grammatically incorrect in the current context, we have emended it to the above form.

1 The Indian mathematical tradition and the place of Bhāskara in it

While Bhāskara was pre-eminent among mathematicians and astronomers, he paid due acknowledgement to his predecessors for the inspiration he derived from them. At the beginning of the *Siddhāntaśiromaṇi* (I.2), he states:

कृती जयति जिष्णुजो गणकचक्रचूडामणिः
जयन्ति ललितोक्तयः प्रथिततन्त्रसद्युक्तयः ।
वराहमिहिरादयः समवलोक्य येषां कृतीः
कृती भवति मादृशोऽप्यतनुतन्त्रबन्धेऽल्पधीः ॥

kṛtī jayati jiṣṇujo gaṇakacakracūḍāmaṇiḥ
jayanti lalitoktayaḥ prathitatantrasadyuktayaḥ |
varāhamihirādayaḥ samavalokya yeṣāṃ kṛtīḥ
kṛtī bhavati mādṛśo'pyatanutantrabandhe'lpadhīḥ ||

Triumphant is the blessed Brahmagupta, the son of Jiṣṇu, who is a crest-jewel among the mathematicians. [Similarly,] exalted are those like Varāhamihira, who have excelled in reasoning out principles of mathematics, and have authored beautiful works (*lalitoktayaḥ*) in mathematical astronomy. Having carefully studied (*samavalokya*) their works, even a lesser intellect like me becomes blessed/qualified (*kṛtī*) to compose large (*atanu*) technical treatises.

By the use of the word *kṛtī*, here a double entendre meaning both 'blessed' and 'qualified', Bhāskara clearly expresses his indebtedness to earlier mathematicians like Brahmagupta and Varāhamihira. Having gained a solid footing by studying their works, Bhāskara made significant contributions to advance the state of knowledge in specific areas of astronomy and mathematics. Thus, he was able to forge ahead and develop many existing discoveries, particularly with reference to planetary phenomena, with his own elegance and eloquence, which left an indelible mark on the Indian astronomical tradition.

From the fifth century onwards, authors including Āryabhaṭa, Brahmagupta, Bhāskara I, Śrīdhara, Mahāvīra, and Śrīpati, all made significant contributions to the advancement of mathematics and astronomy in India. For instance, one of Āryabhaṭa's most brilliant contributions lies in his formulation of a method for finding the first-order sine differences, which is tantamount to solving the harmonic equation $y'' + y = 0$ in its discrete form. He also gave a procedure for solving first order indeterminate equations, known as *kuṭṭaka*. Brahmagupta in the seventh century, carrying forward the tradition, came up with the ingenious law of composition called '*bhāvanā*' for solving second order indeterminate equations, which is considered to be a landmark achievement in the annals of algebra. Some other notable results in mathematics such as the cyclic method (*cakravāla*) for solving equations of the type $x^2 - Dy^2 = 1$ for a

given D with integer solutions for x and y, usually attributed to Bhāskara, can actually be traced to the work of the earlier astronomer Jayadeva. However, it was Bhāskara who brought the *cakravāla* method into prominence through his compositions, casting it in charming and memorable verses.

A particularly significant insight in the Indian tradition was the enunciation of a result equivalent to the differential formula $\delta \sin\theta = \cos\theta\ \delta\theta$. This relation, first articulated by Āryabhaṭa II, was advanced by Bhāskara in the context of discussing the instantaneous motion of planets. It was first brought to the attention of modern scholars by Bapudeva Sastri (1821–1900), while translating *Siddhāntaśiromaṇi* into English along with Lancelot Wilkinson. More broadly, it is interesting to note that the 'derivative' of the sine was extensively employed by Bhāskara in his mathematical formulation of various astronomical problems, with evident awareness that the variable attains its maximum value when its differential vanishes. This of course is not meant to convey that he developed the modern notion of derivative. Nevertheless, it indicates that some fundamental principles in the realm of infinitesimal calculus were discovered and ingeniously applied in the context of corrections to planetary positions.

In geometry, though Śrīpati had given the exact result for the surface area of a sphere prior to Bhāskara, his value of π was less accurate. Moreover, it was Bhāskara who gave the correct formula for both the surface area and the volume of the sphere. In addition, he was the first to have devised and presented the brilliant demonstration of the result by ingeniously summing up the values of canonical sines $\sum_{i=1}^{24} \sin i\alpha \ \ (\alpha = 225')$.

Some of Bhāskara's other notable results include the 'net of numbers' (*aṅkapāśa*) for problems involving permutations and combinations and the general rule for $\sin(A+B)$. He also drew into ever more prominence the importance of 'algebra' (*avyakta-gaṇita*) in mathematics by composing a work dedicated entirely to the subject. In the field of astronomy, it was Bhāskara who seems to have first posed the interesting question concerning why the earth stands unsupported in space, given that all the heavy objects in space fall on the surface of the earth due to its force of attraction. The explanation offered by him may not be quite convincing in modern times, but given the limitations of the theoretical framework that was available for a clear understanding of the phenomena, it is indeed remarkable that he made an attempt to pose such a question and answer it. Again it is Bhāskara who seems to have been the first astronomer to have discussed the concept of *kṣayamāsa* or the omitted lunar month, a subtle and brilliant discovery which does not appear in any of his predecessors' works.

Above all, from a pedagogical view point, he was, as far as we know, the first astronomer to have written an auto-commentary to his works. This com-

mentary, though lucid in style, is succinctly written and is totally devoid of verbosity. Besides helping the reader to easily assimilate the content of his verses, it contributes greatly to interpreting the verses.

2 The works of Bhāskara

In addition to the *Siddhāntaśiromaṇi*, Bhāskara authored the astronomical handbook, the *Karaṇakutūhala*, as well as two mathematical treatises, the *Līlāvatī* and the *Bījagaṇita*, on arithmetic and algebra respectively. Within a short period of their composition, they became canonical expository works in each of these Sanskrit genres, and they still serve to be so after almost nine centuries. However there has been considerable disagreement amongst scholars—both early and modern—regarding the status of the *Līlāvatī* and the *Bījagaṇita*. The issue of contention is whether these two treatises should be considered as independent works or rather as parts of the *Siddhāntaśiromaṇi*, which contains the *Grahagaṇitādhyāya* and the *Golādhyāya*. In the following, we outline some aspects of the history of this claim, as well as argue why it is highly problematic.

2.1 *Traditional views presenting the Līlāvatī and the Bījagaṇita as a part of the Siddhāntaśiromaṇi*

As early as the sixteenth century, discussions concerning the status of these works can be found in the literature. The first instance we know of is from renowned Bhāskara commentator, Gaṇeśa Daivajña, who raises the issue at the beginning of his *Buddhivilāsinī*, a lucid commentary on the *Līlāvatī* [Līlā1937, pp. 4–5]:

> श्रीभास्कराचार्यो ग्रहगणितस्कन्धरूपं सिद्धान्तं चिकीर्षुः ग्रहगणिताद्युपयोगित्वेन तदध्यायभूतं पाटीगणितम् आदावारभते ।
>
> *śrībhāskarācāryo grahagaṇitaskandharūpaṃ siddhāntaṃ cikīrṣuḥ grahagaṇitādyupayogitvena tadadhyāyabhūtaṃ pāṭīgaṇitam ādāvārabhate* |
>
> Bhāskarācārya, desirous of composing a *siddhānta* dealing with sections (*skandha*) on planetary computations, commences, at the beginning, with a chapter as a part of it that deals with arithmetic, since that [topic] will be extremely useful in planetary computations.

Here Gaṇeśa states two things. Firstly, the phrase *tadadhyāyabhūtaṃ* qualified by *pāṭīgaṇitam* indicates that he considers the *Līlāvatī* to be a chapter of the *Siddhāntaśiromaṇi*. Secondly, by the use of the word *ādau*, he notes that it appears right at the beginning of this work.

This point of view is clearly contradicted by another learned commentator on Bhāskara, Munīśvara, who was writing at the beginning of the seventeenth century. Commenting on the colophon, namely:

> इति भास्करीये सिद्धान्तशिरोमणौ लीलावतीसंज्ञः पाट्यध्यायः समाप्त इति।
>
> *iti bhāskarīye siddhāntaśiromaṇau līlāvatīsaṃjñaḥ pāṭyadhyāyaḥ samāpta iti |*
>
> Thus the chapter on arithmetic (*pāṭi*) called the *Līlāvatī* in the *Siddhāntaśiromaṇi* composed by Bhāskara is over.

Munīśvara notes [Ms-11512, f. 251v]:

> भास्करस्याऽयं भास्करीयः भास्करनिर्मिते सिद्धान्तशिरोमणौ ग्रन्थे पाताधिकारानन्तरमारब्धोऽयं पाट्यध्यायो लीलावत्याख्य इत्येवं सम्यक्प्रकारेण प्राप्तः।
>
> *bhāskarasyā'yaṃ bhāskarīyaḥ bhāskaranirmite siddhāntaśiromaṇau granthe pātādhikārānantaramārabdho'yaṃ pāṭyadhyāyo līlāvatyākhya ityevaṃ samyakprakāreṇa prāptaḥ |*
>
> The word *Bhāskarīya* means that which belongs to Bhāskara. [That is,] in the text *Siddhāntaśiromaṇi* that is composed by Bhāskara, this chapter on arithmetic called the *Līlāvatī*, begun immediately after the chapter dealing with the nodes (*pātas*), has thus been obtained (i.e., completed) properly.

The elaboration made by Munīśvara on the colophon evidently points to the fact that he too believes the *Līlāvatī* to be a chapter of the *Siddhāntaśiromaṇi*. But what is noteworthy here is where he situates this chapter; he is clearly of the opinion that it appears *after* the chapter entitled *pātādhikāra* ('chapter on *pātas*') of the *Grahagaṇitādhyāya*, and not at the beginning, as stated by Gaṇeśa. Thus, though Gaṇeśa and Munīśvara are of the view that the *Līlāvatī* forms a part of the *Siddhāntaśiromaṇi*, there is no agreement between them as to where it appears in the treatise. This confusion among them may point to an underlying unreliability in the textual tradition they had inherited and may indicate subsequent collating efforts of Bhāskara's works by zealous scribes. Indeed, placing the *Līlāvatī* in either of the proposed locations in the *Siddhāntaśiromaṇi* leads to serious problems, as we will show below.

Modern scholars appear to have continued to uphold this claim, seemingly influenced by the testimony of these earlier commentators. Colebrooke, for instance, maintains [Col2005, p. ii]:

> The treatises in question, which occupy the present volume, are the *Víjagańita* and *Lílávatí* of Bháscara Áchárya and the *Gańitád'hyaya* and *Cuttacád'hyaya* of Brahmegupta. The two first mentioned constitute the preliminary portion of Bháscara's

> Course of Astronomy, entitled *Sidd'hántaśirómańi*. The two last are the twelfth and eighteenth chapters of a similar course of astronomy, by Brahmegupta, entitled *Brahma-sidd'hánta*.

It is unambiguous from the above quote, that Colebrooke is of the view that the *Līlāvatī* and the *Bījagaṇita* constitute the early part of the *Gaṇitādhyāya* of the *Siddhāntaśiromaṇi*—precisely the opinion expounded by Gaṇeśa. Continuing further, Colebrooke concludes his discussion on the above topic with the following note [Col2005, p. iii]:

> *Carańa-cutuhala*, a practical astronomical treatise, the epoch of which is 1105 Saca; 33 years subsequent to the completion of the systematic treatise. The date of the *Sidd'hánta-śirómańi*, of which the *Víja-gańita* and *Lílávatí* are parts, is fixt then with the utmost exactness, on the most satisfactory grounds, at the middle of the twelfth century of the Christian era, A. D. 1150.

thus reiterating his position that the two form a part of the larger treatise *Siddhāntaśiromaṇi*. Colebrooke further consolidates his opinion in a footnote [Col2005, p. iii, fn. 3]:

> Though the matter be introductory, the preliminary treatises on arithmetic and algebra may have been added subsequently, as is hinted by one of the commentators of the astronomical part (*Vártic.*). The order there intimated places them after the computation of planets, but before the treatise on spherics; which contains the date.

In this footnote, Colebrooke includes the abbreviation '*Vartic*' in parentheses, which alludes to a gloss by Nṛsiṃha Daivajña. Essentially, Colebrooke is referring to the opinion of Munīśvara. Since this view has been relegated to a footnote, it may suggest that Colebrooke was more inclined to adopt the view of Gaṇeśa over that of Munīśvara.[3] Subsequently, many other scholars, including S. B. Dikshit, K. V. Sarma, B. V. Subbarayappa, who produced either a translation and/or notes on the *Siddhāntaśiromaṇi*, by and large have accepted either of the two positions indicated above. In addition, a few of these scholars seem to have further confounded the issue by arguing that the *Golādhyāya* was intended to come before the *Gaṇitādhyāya*. This view seems to have originated from the fact that Bhāskara, in a few places in his *Vāsanābhāṣya* on the *Gaṇitādhyāya*, cites excerpts from the *Vāsanābhāṣya* on the *Golādhyāya*.

The view that the *Līlāvatī* and the *Bījagaṇita* are part of the *Siddhāntaśiromaṇi* has found its way into more mainstream literature as well. For instance, George Joseph notes in one of his recent works [Jos2016, p. 249]:

[3] It is quite probable that Colebrooke got an idea of Nṛsiṃha's view through some informants.

> His fame rests on *Siddhantasiromani* (Crown of treatises) a work in four parts: *Lilavati* (arithmetic), *Bijaganita* (algebra), *Goladhyaya* (chapter on celestial globe) and *Grahaganita* (the mathematics of planets). The first two parts are essentially texts on mathematics and the last two relate to astronomy.

The sequence of the four parts in the order *Līlāvatī*, *Bījagaṇita*, *Golādhyāya*, and *Grahagaṇitādhyāya* proposed by Joseph shows he too subscribes to the view that the *Golādhyāya* was intended by Bhāskara to precede the *Grahagaṇitādhyāya.*

2.2 Compelling reasons to consider the *Līlāvatī* and the *Bījagaṇita* as independent works

There are several problems with the claim that the *Līlāvatī* and *Bījagaṇita* form a part of the *Siddhāntaśiromaṇi.* These arise when one considers features and excerpts from the texts themselves. The first and most significant is that nowhere in the works of Bhāskara do we find him expressing the view that the *Siddhāntaśiromaṇi* consists of four parts. Neither do we find pointers indicating this four-fold division in his auto-commentary, the *Vāsanābhāṣya.* In fact, there is much evidence that points to the fact that the *Vāsanābhāṣya* was authored much later than the *Siddhāntaśiromaṇi*, the *Līlāvatī*, and the *Bījagaṇita.*

Furthermore, assuming that the *Līlāvatī* was a part of the *Siddhāntaśiromaṇi*, it seems appropriate, given the contents of this text, that it would appear in the early part of the treatise, and not somewhere later on. However, it cannot be placed in the early part of the treatise, before, say, the *Gaṇitādhyāya*, since the third verse of the *Gaṇitādhyāya*, '*kṛtvā cetasi bhaktito …*',[4] provides the title of the *Siddhāntaśiromaṇi.* If the *Līlāvatī* was to come before the *Gaṇitādhyāya*, there would be several hundred verses before this third verse wherein Bhāskara puts forth his intention of composing the *Siddhāntaśiromaṇi.* It would be extremely unusual for an author to present the title of a work so late in the treatise.

There are additional reasons as to why the view that the *Līlāvatī* and *Bījagaṇita* constitute two chapters in between the *Gaṇitādhyāya* and the *Golādhyāya* isn't tenable. Notably, there is no *Vāsanābhāṣya* for the *Līlāvatī* or the *Bījagaṇita*, as there are for the *Gaṇitādhyāya* and the *Golādhyāya.* It would be very peculiar for Bhāskara, who is so eloquent and elaborate in his *Vāsanābhāṣya*, to restrict his auto-commentary to merely two of the four

[4] For the full verse see p. i.

chapters of the *Siddhāntaśiromaṇi* (the proposed first and the last chapters) and simply disregard the other two chapters, providing only the occasional brief explanation and illustrative examples.

Moreover, the verses that appear in the concluding part of both the *Līlāvatī* and *Bījagaṇita* seem to sustain the notion that they are meant to be independent treatises. For instance, in the *Bījagaṇita*, we find the following verse appearing towards the very end of the work (verse 217):

आसीन्महेश्वर इति प्रथितः पृथिव्यां आचार्यवर्यपदवीं विदुषां प्रपन्नः ।
लब्ध्वाऽवबोधकलिकां तत एव चक्रे तज्जेन बीजगणितं लघु भास्करेण ॥

āsīnmaheśvara iti prathitaḥ pṛthivyāṃ
ācāryavaryapadavīṃ viduṣāṃ prapannaḥ |
labdhvā'vabodhakalikāṃ tata eva cakre
tajjena bījagaṇitaṃ laghu bhāskareṇa ||

There was a great scholar,[5] well-known to the world, by name Maheśvara, who achieved the distinction of being described as foremost amongst the preceptors. This shorter version of the *Bījagaṇita* was composed by his son, Bhāskara, having gained just a little of his knowledge.

Here, Bhāskara describes what a great teacher and scholar his father Maheśvara was, and that he has just shared a part of his knowledge (*bodhakalika*) that he could gain from him. Traditionally, such references to one's own lineage, and that of one's teachers, are done only at the end of treatises, and not anywhere in between. If the *Bījagaṇita* were to be a middle chapter of the *Siddhāntaśiromaṇi*, why would Bhāskara describe his lineage there, which would fall somewhere in the middle of the large treatise? Such a feature would put him at odds with well-established standard practices and norms in acknowledgements that appear towards the end of a treatise.

There are additional compelling passages from the two treatises which further reinforces the idea that these were independent treatises. A careful reading of the style of the *upakrama* (beginning note) and the *upasaṃhāra* (concluding note), that appear in these two treatises, certainly compels one to think that they are independent treatises themselves. Towards the end of the *Bījagaṇita*, we find the following statement (verse 224):

तथा गोलाध्याये मयोक्तम् --
अस्ति त्रैराशिकं पाटी बीजं च विमला मतिः ।
किमज्ञातं सुबुद्धीनामतो मन्दार्थमुच्यते ॥

[5] The use of the word *āsīt*, notably past tense, is clearly indicative of the fact that Bhāskara's father and teacher, Maheśvara, was not alive during the period of composition of this work. The use of an almost identical phrase in *Praśnādhyāya*, verse 61, towards the end of the *Siddhāntaśiromaṇi*, reveals that Bhāskara had lost his father before he was 36.

tathā golādhyāye mayoktam –
asti trairāśikaṃ pāṭī bījaṃ ca vimalā matiḥ |
kimajñātaṃ subuddhīnāmato mandārthamucyate ||

In a similar way, it was stated by me in the *Golādhyāya*:
For those intelligent ones who have [a grasp of] the rule of three [which is essentially] arithmetic (*pāṭī*) and pure intellect [which is essentially] algebra (*bīja*, literally seed or primary cause), what is unknown? Accordingly, [this work] is set forth for the slow-witted.

This is fairly telling. A statement as above cannot be made by an author unless the text they are referring to was already completed; in this case, the *Golādhyāya* must have been completed before Bhāskara authored the *Bījagaṇita*. If this were not the case, the statement preceding the verse quoted above would be meaningless; alluding to a verse which has not yet been composed as "already done" is impossible. Indeed, almost all editions of the *Bījagaṇita* include this statement. This is but one of several instances in the text which refute the opinion of the large single four-chaptered work.

There is another instance of convincing evidence from Bhāskara's own statements that point to the idea that the *Siddhāntaśiromaṇi* was authored before he wrote the *Bījagaṇita* (along with its brief explanations in prose). While commenting on verses 37 and 38 of the *Bījagaṇita*, Bhāskara notes:

अस्योदाहरणानि प्रश्नाध्याये ।[6]

asyodāharaṇāni praśnādhyāye

The examples of this are [to be found] in the Question-chapter.

Since the *praśnādhyāya* referred to in the above quotation forms the last section of the *Golādhyāya* of the *Siddhāntaśiromaṇi*, how could one explain the above statement by presuming the *Bījagaṇita* to be an earlier part of the *Siddhāntaśiromaṇi*? This of course can be explained by supposing that sections of the *Bījagaṇita* and brief explanatory notes were supplied by Bhāskara later. But such a supposition seems unlikely.

In light of the arguments raised above, it seems more compelling to regard the *Līlāvatī* and the *Bījagaṇita* as independent works of Bhāskara—as we consider the *Karaṇakutūhala* to be so—and not as an integral part of *Siddhāntaśiromaṇi*. One might also question, as has been done in the case of the *Līlāvatī* and the *Bījagaṇita*, the status of the *Golādhyāya* as being an independent work also. However this appears very unlikely, since one of the most famous verses in the *Siddhāntaśiromaṇi*, which gives its date of composition, appearing in the *Praśnādhyāya* ('Chapter on [the three] questions') of the

[6] See [BīGa2009, p. 31], wherein Hayashi has also supplied in parentheses '*golādhyāyāntargata*'.

Golādhyāya, seems definitive. We find the word *Siddhāntaśiromaṇi* appearing in the last quarter of the verse (*Praśnādhyāya* 58):

रसगुणपूर्णमहीसमशकनृपसमयेऽभवन्ममोत्पत्तिः ।
रसगुणवर्षेन मया सिद्धान्तशिरोमणी रचितः ॥

rasaguṇapūrṇamahīsamaśakanṛpasamaye'bhavanmamotpattiḥ |
rasaguṇavarṣena mayā siddhāntaśiromaṇī racitaḥ ||

My birth took place in 1036 of the reign reckoned after the Śaka kings. When I was 36, the *Siddhāntaśiromaṇi* was composed by me.

just as it appeared right at the beginning, in verse 3, of the *Gaṇitādhyāya*.

As regards to the sequence of composition, it seems plausible that given the various cross references through out the text, the following sequence of works does not lead to any contradiction:

1. *Līlāvatī*
2. *Siddhāntaśiromaṇi* (*Gaṇitādhyāya* and *Golādhyāya*)
3. *Bījagaṇita*
4. *Vāsanābhāṣya*[7]
5. *Karaṇakutūhala*

Indeed, these works served as an inspiration for generations of scholars to come. Future astronomers grappled with the nuances of Bhāskara's rules, but were also motivated to develop and improve on them. Even in modern times, his works have widespread appeal and still continue to charm. Many of his versified techniques and procedures have achieved legendary status and are used in the classroom to add historical interest to teaching mathematics as much as they are held up as exemplars of the heights to which Indian thinkers have been able to achieve. Accordingly, aspects and elements from his works have been explored by modern scholars whose reflections fill the chapters of this book.

3 Synopses of the contributions

This volume is divided into seven parts to reflect the various areas that modern scholarship on Bhāskara and his contributions has focused on. At the

[7] Even within the *Vāsanābhāṣya*, there are pointers to show that the commentary to the *Golādhyāya* should precede that of *Gaṇitādhyāya*.

beginning of each part, a *maṅgalācaraṇa* (opening invocatory verse) from the appropriate work of Bhāskara is given in Sanskrit along with a poetic translation. The volume ends with a substantial bibliography which is common to all chapters, and is divided into three parts: Primary sources, Secondary sources, and Catalogues, dictionaries, and manuscripts. Its length testifies to the considerable amount of scholarship Bhāskara's works have inspired over almost nine centuries. In the following, a brief synopsis of each chapter is presented.

3.1 Part I: Bhāskarācārya and his poetic genius

Details surrounding the lineage and circumstances of ancient authors in India are usually conspicuously absent in the historical record, even for very famous, prolific scholars. However, historians are in a more fortunate position when it comes to Bhāskara, due to the existence of two stone inscriptions that provide much information concerning the life and ancestry of this mathematician, as well as testify to his excellent reputation and celebrate his extraordinary scholarship. The very first contribution of this volume, *Caṅgadeva's Inscription of 1207*, includes the text and translation of one of these inscriptions, along with an account of its discovery and study by Indian and European archaeologists, supplemented by an analysis of its contents. The two inscriptions are still extant today and may be viewed at their respective sites. To appreciate their current state and locality, the editors have provided photographic documentation from recent visits to these sites along with oral testimony from the local temple officials, whose families have been connected to the site for generations.

One of the most persistent stories in Indian mathematical lore is the so-called legend of *Līlāvatī*. This legend, which has since assumed the status of a well-known fact in many historical accounts, is explored by S. R. Sarma in his contribution *The Legend of Līlāvatī*. Multiple versions of the story circulate but all generally include the theme that *Līlāvatī* was Bhāskara's daughter and by misfortune, was either widowed or unable to be married. As a consolation, Bhāskara wrote a book on arithmetic and named it after her. Locating the earliest known instance of the legend in a Persian translation appearing in 1587 by poet Abū'l Fayḍ Fayḍī, Sarma then systematically evaluates the evidence to reveal that the legend has undoubtedly no substance and was probably made up to explain the curious title of the book. To complete the investigation, Sarma explores the origin of the legend and looks at stories which have a similar nature. The lessons learned for historians are important ones. As Sarma observes: "It is one of the tasks of the history of science to

dispel the legends and establish the true facts of the life and achievements of the scientists of the past ages".

The poetic genius of Bhāskara has long been recognised and explains the enduring appeal of his works above and beyond their scientific content. To this end, Pierre Filliozat combines literary analysis with due acknowledgement to the mathematical content to highlight the specific literary devices Bhāskara frequently employs in his contribution *The poetical face of the mathematical and astronomical works of Bhāskarācārya*. With recourse to the discipline of *alaṅkāraśāstra*, Filliozat discusses how a state of *rasa* or 'aesthetic savour' is evoked in the reader. In doing so, he details the poetic embellishments employed by Bhāskara by excerpting several examples from his works and demonstrates how successful he was in bringing a fusion between science and poetry, eliciting delight in his readers.

3.2 Part II: The Līlāvatī

In a similar spirit, the contribution of K. Ramasubramanian, K. Mahesh, and Aditya Kolachana, entitled *The Līlā of the Līlāvatī*, explores reasons for the perennial charm of the *Līlāvatī*, by simultaneously highlighting the technical and literary features of many of its verses. They reveal hitherto unexplored aspects of the text such as Bhāskara's balancing of brevity and clarity in his verses, the imaginative qualities of his examples, the variety of metres he employs, and the range of *alaṅkāra* that he uses to enhance the appeal of his work. By presenting numerous examples from various sections of the *Līlāvatī*, they affirm Bhāskara's mastery of both the poetic arts and the technical sciences, which justifies his reputation as an author of unique and unparalleled significance.

The *Līlāvatī* was extremely popular and among the many copies made of this text, a significant proportion circulated in scripts other than Devanāgarī. In their contribution *Two Malayalam commentaries on the Līlāvatī*, N. K. Sundareswaran and P. M. Vrinda detail the existence of commentaries in Malayalam script on the *Līlāvatī*. They explore these two commentaries, which they call the *Yogāśraya* and the *Abhipreta*, from a selection of nine manuscripts. In their analysis, they offer direct excerpts from the commentaries revealing that these Malayalam commentators often offered alternative methods to those given in the *Līlāvatī*, which were presumably intended to supplement the content of the text, or add value to their own composition or even offer better and varied options to the reader. In particular, the methods outlined by them for the extraction of the square root and the cube root of a

number are noteworthy. Accordingly, they give us a rich sense of the textual tradition of the *Līlāvatī* in the centuries after its composition, and the ways in which it circulated in various regional cultures of inquiry as well as the means by which practitioners continued to keep old texts relevant by blending them with newer techniques and procedures.

Another instance of the reception of the *Līlāvatī* is explored by K. Ramakalyani in her contribution entitled *Gaṇeśa Daivajña's upapattis for some rules of the Līlāvatī*. Here, she gives an account of select *upapattis* from the one hundred or so Gaṇeśa includes in his commentary, the *Buddhivilāsinī*. Ramakalyani identifies some common themes in these *upapattis*; some take the form of a logical or verbal explanation, others include algebraic proofs, demonstrations, geometric proofs, and the like. For each of these modes of exegesis, she chooses a salient example, citing the relevant excerpt from the *Līlāvatī* and the passage from Gaṇeśa's commentary along with an explanation and discussion of the content. By doing so, she gives insight into the sorts of features that caught the commentators eye and their response with various appropriate modes of reasoning.

The *Līlāvatī* must also be understood in a broader context, namely in a long established tradition of arithmetical texts containing mathematical rules, procedures, and examples. One particular topic that enjoyed notable attention was that concerning the areas of trilaterals and quadrilaterals. In his contribution *Mensuration of quadrilaterals in the Līlāvatī*, S. G. Dani surveys the various rules given by Indian authors predating Bhāskara, including Brahmagupta, Śrīdhara, Mahāvīra, and Śrīpati. Against this backdrop he examines Bhāskara's treatment of the topic, both in his verse-text and in his auto-commentary and explores Bhāskara's concern for a perceived flaw in the traditional formulas. In this way he illustrates the ways in which later mathematicians engaged with their predecessors over a mathematical issue both heuristically as well as logically.

While being part of a continuous tradition in mathematics on the Indian subcontinent, it is not uncommon to find mathematicians introducing new rules to solve old problems as well as new topics. One such topic that Bhāskara introduces in his *Līlāvatī* is one called *aṅkapāśa* ('net of numbers'). This is explored in detail by Takanori Kusuba in his contribution entitled *Aṅkapāśa in the Līlāvatī*. This topic considers various permutations of digits which does not appear in any of Bhāskara's predecessors' works. Kusuba examines four rules concerning such permutations with examples along with commentaries on these passages by Bhāskara himself, and by Gaṇeśa and Nārāyaṇa.

3.3 Part III: The Bījagaṇita

While the *Līlāvatī* concerns arithmetic, Bhāskara's companion work, the *Bījagaṇita*, focuses on algebra. An overview of this work is provided by Sita Sundar Ram in her contribution *The Bījagaṇita of Bhāskarācārya.* From the definition of *bīja*, through notions of zero and infinity, to indeterminate equations and quadratics, she gives us an insight into some of the mathematical highlights contained in this text. In her discussion on the problem related to *vargaprakṛti*, she includes an interesting historical note, relating to how Fermat posed the same problem as a challenge to his fellow mathematicians, obviously unaware that it had been solved in India almost six centuries earlier. In this way, she showcases the variety of topics included in this genre and the detailed way in which Bhāskara expounded on them.

Among the many topics it addresses, the *Bījagaṇita* devotes a chapter to rules related to surds entitled *karaṇīṣaḍvidham* ('six operations on the surd'). Shriram M. Chauthaiwale examines and analyses these rules in his contribution *The critical study of algorithms in Karaṇīṣaḍvidham.* He provides a translation of the 22 verses that make up this chapter, along with an explanation of the algorithms they contain. With these explanations and numerical examples, Chauthaiwale highlights the novel contributions of Bhāskara on this subject, namely, his observance of the conditions under which these rules work, as well as their limitations and the various criteria according to which a solution can be found.

3.4 Part IV: The Siddhāntaśiromaṇi: Gaṇitādhyāya

Besides the two works devoted to mathematics, Bhāskara wrote the monumental *Siddhāntaśiromaṇi*, an astronomical text providing astronomers with detailed procedures and parameters for the purpose of determining planetary motion, eclipses, calendrics and time-keeping. This treatise is also accompanied by an auto-commentary, called the *Vāsanābhāṣya*, in prose. In his contribution entitled *Grahagaṇitādhyāya of Bhāskarācārya's Siddhāntaśiromaṇi*, M. S. Sriram selects various key results from the early chapters concerning mean and true planetary motion and locating direction and time. In order to illustrate the ways in which Bhāskara explained and rationalised the rules in the *Siddhāntaśiromaṇi*, Sriram excerpts pertinent passages from Bhāskara's commentary and provides a translation and detailed technical analyses of their contents using modern notation. His explanatory notes, closely following Bhāskara's *Vāsanābhāṣya*, enable the reader to have a full and direct

appreciation of the voice of Bhāskara wherein he not only presents the rules, but articulates what motivated him to formulate them.

A critical part of the efforts of astronomers was the regulation of the calendar. A notable feature of the Indian calendar was the practice of omitting a lunar month, called *kṣayamāsa*, to keep lunar and solar phenomena in sync. In his contribution, *Bhāskarācārya and kṣayamāsa*, Michio Yano studies this feature, observing that the first astronomer to explicitly discuss this problem was Bhāskara. Examining passages from the *Gaṇitādhyāya* of the *Siddhāntaśiromaṇi*, Yano carefully explains Bhāskara's rules and parameters from first principles. Furthermore, to understand better the frequency of these omitted months and how accurate Bhāskara's rules were, he makes a series of comparisons with results generated by his *pañcāṅga* program. In doing so, Yano touches upon the fascinating issue of theory and practice within astronomy in second millennium Sanskrit sources.

3.5 Part V: The Siddhāntaśiromaṇi: Golādhyāya

Besides his detailed auto-commentary the *Vāsanābhāṣya* to the *Siddhāntaśiromaṇi*, Bhāskara also provided short notes to other mathematical works namely the *Līlāvatī* and the *Bījagaṇita*. M. D. Srinivas more broadly considers Bhāskara's role as a commentator in various other works in his contribution *Vāsanābhāṣyas of Bhāskarācārya*. In doing so, he compares and contrasts Bhāskara's exegetical style with those of other, later commentators on similar passages, offering lucid yet precise English translations of the long passages in Sanskrit. Among the topics he covers are a discussion on the precession of the equinoxes, the variation of astronomical parameters in different texts, and the rationale behind various guiding principles. Through this, Srinivas gives us a glimpse of the impact Bhāskara had on later generations of astronomers and the ways in which these later thinkers grappled with and resolved issues in the astronomical sciences.

Indispensable to astronomy is trigonometry, and rules pertaining to sines and related trigonometric functions. These rules are generally found near the beginning of a *siddhānta* treatise, most commonly in the chapter devoted to determining the true longitudes and motions of the planets. In addition to a comprehensive account of sine values, Bhāskara allocated an entire separate chapter to the topic of trigonometry, entitled the *Jyotpatti*, thus being one of the first scholars to treat this topic in a somewhat independent way. In her contribution *The production of sines in Bhāskarācārya's Jyotpatti*, Clemency Montelle examines this chapter in detail. Only 25 verses in length, Montelle

analyses the mathematical relations included in this chapter for determining various sine values, above and beyond the 24 values Bhāskara includes earlier in the work. Invoking passages from the text, she speculates why Bhāskara provides multiple rules, considers the technical terminology associated with these trigonometric procedures, and explores how this special treatment underscores an increasing interest to treat trigonometry outside the context of astronomy, as an intrinsically interesting and important mathematical topic in its own right.

One of the notable rules included in the *Jyotpatti* is the so-called sums to products formula involving sines and cosines. In fact, this is the first time this rule was explicitly stated in the Indian tradition. Setsuro Ikeyama examines this rule and its justification in his contribution *An application of the addition and subtraction formula for the sine in Indian astronomy.* Ikeyama then further considers its treatment and rationalisation by later scholars, Mādhava and Nīlakaṇṭha, who derive the rule in several different ways and explicitly link it to various astronomical phenomena using plane trigonometry in the sphere. Ikeyama thus not only reveals how the rules were arrived at, but also demonstrates how they underpin many complicated astronomical expressions.

It is inconceivable indeed to think of an astronomical tradition that does not make regular observations aided by instrumentation, however crude they may be as compared to the sophisticated devices we may have today. In his contribution *Astronomical instruments in Bhāskarācārya's Siddhāntaśiromaṇi*, S. R. Sarma takes to task claims that while Bhāskara advanced many important mathematical and astronomical procedures, he did not make much headway with respect to observational astronomy. He does this by a through study of the *Siddhāntaśiromaṇi*'s chapter devoted exclusively to astronomical instruments, the *Yantrādhyāya.* As well as detailing the contents of this chapter, he places Bhāskara firmly in an evolving tradition of descriptions of observational instruments beginning with Brahmagupta and explores Bhāskara's attitude towards observational astronomy using key passages from this chapter.

3.6 Part VI: The Karaṇakutūhala

The other seminal astronomical work Bhāskara composed was the *karaṇa* text entitled *Karaṇakutūhala.* Some of the ingenious approximating simplifications, typical of a *karaṇa* work, are explored by Kim Plofker in her contribution *Bhojarāja and Bhāskara: Precursors of Karaṇakutūhala algebraic approximation formulas in the Rājamṛgāṅka.* In particular, Plofker considers formulas

which involve algebraic approximations for measuring gnomon shadows and the possible connections with those found in the earlier handbook, the *Rājamṛgāṅka* of Bhojarāja. Through a careful textual comparison and detailed mathematical analyses she explores issues relating to the ways in which these masterly approximations might have been made.

3.7 Part VII: Reach of Bhāskara's work and its pedagogical significance

Bhāskara's works circulated widely and their popularity grew beyond regional boundaries to different cultures of inquiry. In particular, many of his works were translated into Persian as part of a larger translation movement during the Mughal period. S. M. Razaullah Ansari documents this transmission in his contribution *Persian translations of Bhāskara's Sanskrit texts and their impact in the following centuries.* In his account he considers the motivations for commissioning translations, the circumstances of specific translators and their repertoire, and examines manuscript surveys of various repositories holding translations of the works of Bhāskara. He then outlines the impact Bhāskara's work had on scientific traditions within the Mughal community as well as exploring more generally the phenomenon of translations and how they impacted upon the inheritor culture's astronomical and mathematical practices.

The challenges of weaving mathematical and astronomical content into verse are numerous, not least is the task of encoding long strings of numbers key to these subjects into a metrical framework. One of the earliest systems devised by Indian mathematicians to achieve this was called *Bhūtasaṅkhyā*, or the object-numeral system of numeration. Medha Shrikant Limaye in her contribution entitled *Use of Bhūtasaṅkhyā in the Līlāvatī of Bhāskarācārya* surveys the use of this system and the ways in which Bhāskara takes advantage of it to fulfil his poetic aspirations. Against a backdrop of the historical uses of this system, Limaye collects all the instances of *Bhūtasaṅkhyā* used in the *Līlāvatī* and classifies these into various branches of inquiry, including astronomy, calendrics, philosophy, and so on, with an explanation and citation of the textual passage where appropriate to clarify where the numerical associations of these words derive from.

The use of the *Līlāvatī* as an introductory text to teach mathematics bears testimony to its clarity and charm, and indeed that it continues to capture modern audiences almost a millennium after it was composed is further confirmation of its appealing character. Hari Prasad Koirala explores how and

why this feature of the *Līlāvatī* is worth exploring for pedagogical purposes in his contribution *Implications of Bhāskarācārya's Līlāvatī to the common core state standards in mathematics.* He highlights the specific ways in which the *Līlāvatī* can be used to enhance some of the common core state standards in the current United States mathematics curriculum, including topics in arithmetic, elementary algebra, geometry, and mensuration. Among the many examples he offers, Koirala highlights the ways in which Bhāskara's verses can not only inspire students with their subject matter, but help them use tools to enhance their mathematical understanding as well as their ability to communicate their ideas.

4 Outreach programmes

In addition to scholarly contributions, the 900th birth year of Bhāskara was celebrated by developing and facilitating various outreach programs targetting students of all ages throughout India to showcase and honour his name and achievements. The ambitions of this grand scale outreach project was to instil a sense of pride in younger generations of one of their most celebrated thinkers, by sketching some details of his mathematical achievements, and capturing their imagination through his poetic flair.

The individual who championed this vision was Dr. Vijay Bedekar, and it was his colleague Prof. Sudhakar Agarkar, who in a span of this single calendar year, conducted sixty-five workshops at different places throughout the states of Maharashtra, Andhra Pradesh and Madhya Pradesh, reaching thousands of students. The resulting outreach was thus remarkable. These workshops were arranged for students studying at many different levels. Workshops were also extended to reach students undergoing professional courses, such as those enrolled in polytechnics, and engineering or management programs. In addition, arrangements were made so that teachers, parents, and members of industrial associations could attend dedicated workshops too. On the request of many educational institutions in India, further workshops on the *Līlāvatī* have been conducted beyond 2014.

The general format of the workshop was comprised of two sessions of 90 minutes each. The first session was devoted to acquainting the participants with the long and rich tradition of mathematics in India. Contributions of ancient thinkers such as Baudhāyana, Piṅgala and Kātyāyana were discussed, followed by the contributions of early mathematicians like Āryabhaṭa and Brahmagupta who flourished in the middle of the first millennium. Against this historical backdrop, Bhāskara was introduced and the influence of previ-

ous mathematicians on his work, as well as his innovations, were brought out clearly. Reference was also made to the inscription by his grandson Caṅgadeva at Pāṭanādevī near Chalisgaon in Maharashtra (see Part 1).

The second session of the workshop was devoted to Bhāskara's famous book on arithmetic, the *Līlāvatī*. The salient features of this composition, including its poetic descriptions, its didactic nature, its simplicity and clarity, and so on, were highlighted. Students were exposed to Bhāskara first hand as they were presented with mathematical problems directly excerpted from the *Līlāvatī* and were asked to solve them. The choice of the problems was made carefully keeping in mind the level of the abilities of the audience. From simple problems involving basic arithmetical operations to more sophisticated ones which involved trigonometry, progressions, or interest computations, students of all levels could enjoy problems from Bhāskara.

The experience of conducting these workshops has been very encouraging. Students delighted in reading this mathematical text book and tackling the practical problems Bhāskara posed. It gave them an appreciation of historical mathematics in its own context, and also allowed them to enjoy solving problems because of the timeless appeal of Bhāskara's way of formulating the problems as attractive examples. An important outcome of the workshops on the *Līlāvatī* was raising a greater awareness among students and teachers of India's rich mathematical heritage. While most historical surveys focus on a somewhat Eurocentric account of mathematical achievements, these workshops enabled the participants to realise that India's contribution to mathematics has been substantial. More importantly, Bhāskara is now a familiar name to the many thousands of students who participated in these workshops.

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Clemency Montelle

Part I

Bhāskarācārya and his poetic genius

उद्भटबुद्धिर्भाट्टे सांख्येऽसंख्यः स्वतन्त्रधीस्तन्त्रे ।
वेदेऽनवद्यविद्योऽनल्पः शिल्पादिषु कलासु ॥
स्वच्छन्दो यश्छन्दसि शास्त्रे वैशेषिके विशेषज्ञः ।
यः श्रीप्रभाकरगुरुः प्राभाकरदर्शने कविः काव्ये ॥
बहुगुणगणितप्रभृतिस्कन्धत्रितये त्रिनेत्रसमः ।
विबुधाभिवन्दितपदो जयति श्रीभास्कराचार्यः ॥

udbhaṭabuddhir bhāṭṭe sāṃkhye 'saṃkhyaḥ svatantradhīs tantre |
vede 'navadyavidyo 'nalpaḥ śilpādiṣu kalāsu ||
svacchando yaś chandasi śāstre vaiśeṣike viśeṣajñaḥ |
yaḥ śrīprabhākaraguruḥ prābhākaradarśane kaviḥ kāvye ||
bahuguṇagaṇitaprabhṛtiskandhatritaye trinetrasamaḥ |
vibudhābhivanditapado jayati śrībhāskarācāryaḥ ||

Triumphant is the illustrious Bhāskarācārya, whose feet are revered by the wise, [who was] eminently learned in Bhaṭṭa's doctrine, unique in the *Sāṁkhya*, an independent thinker in the *Tantra*, possessed of unblemished knowledge of the *Veda*, [and] inferior to none in sculpture and other arts, who could employ [various] poetic metres at his will, who was deeply versed in the *Vaiśeṣika* system, who was the illustrious Prabhākaraguru in Prabhākara's doctrine [of Mīmāṃsā], who was the poet in [the art of writing] poetry, and who was equal to the three-eyed [God Śiva] in the three branches [of *Jyotiḥśāstra*], beginning with *gaṇita* which comprises various [mathematical] procedures.

Caṅgadeva’s inscription of 1207

Editors*

1 Discovery and study of the inscription

Bhāskarācārya’s grandson Caṅgadeva established a college (*maṭha*) for the propagation of his grandfather’s works (*śrī-bhāskarācārya-nibaddha-śāstra-vistāra-hetoḥ*) and announced this fact in an inscription in 1207. This inscription was discovered by the noted antiquarian scholar Dr. Ramachandra Vittal Lad, popularly known as Bhau Daji (1822–74), some time before 1865 and published in that year [Daj1865, pp. 392–418]. Later, when the journal *Epigraphia Indica* was launched, F. Kielhorn published, in the very first volume of this journal, the text of the inscription together with his own English translation in a format which became the standard style[1] for the publication of inscriptions in India [Kie1892, pp. 338–346]. Since this inscription throws valuable light on the family of Bhāskarācārya, the text of the inscription is reprinted in this volume dedicated to Bhāskarācārya, together with Kielhorn’s English translation.[2] The inscription was composed in Sanskrit verse, but the operative part, which details the obligations of the public, is couched in a form of the contemporary local variant of old Marathi so that it is accessible to the common people. Kielhorn did not translate this part. In his linguistic study of old Marathi inscriptions, Alfred Master provided an English translation of this part [Mas1957, pp. 430–431], which is also added here.

Kielhorn states that the inscription is engraved ‘on a stone-tablet in the ruined temple of the goddess Bhavānī at Pāṭṇā, a deserted village about ten

* The editors would like to gratefully acknowledge the inputs received from Prof. S. R. Sarma in preparing this note.

[1] One of the important features of this style is that the text of the inscriptions is reproduced according to the lines in which the inscription is engraved on the stone or on the copper plate.

[2] For the convenience of the readers of this volume, the text is reproduced here according to the metrical lines of Sanskrit verses, without the palaeographic notes added by Kielhorn and the transliteration is modified. The Sanskrit verses presented here are taken from the *Gaṇakataraṅgiṇī* of Sudhakara Dvivedi [GaTa1933, pp. 39–42].

K. Ramasubramanian et al. (eds.), *Bhāskara-prabhā*, Sources and Studies in the History of Mathematics and Physical Sciences, https://doi.org/10.1007/978-981-13-6034-3_1

miles south-west of Chalisgaon in Khāndeś ... It consists of 26 lines which cover a space of about 2 ′6″ broad by 1 ′6″ high. The writing is on the whole well preserved, but at the top a small portion of the surface of the stone has peeled off, causing the nearly complete loss of about a dozen *aksharas* in the middle of the first line; and a few *aksharas* are illegible in the concluding lines. The average size of the letters is half-an-inch. The characters are Nāgarī of about the thirteenth century.'

2 Current status of the inscription

Having gathered information from the officials of the Archaeological Survey of India (ASI) that the inscription is still in the temple of 'Bhavānī', locally known as Pāṭaṇādevī (20°20′ N; 74°58′ E), near the town Chalisgaon, an expedition[3] was made recently to the temple to personally inspect the inscription and take stock of its current condition. Figures 1 and 2 depict the main temple hall (*maṇḍapa*) and a section of the exterior part of the temple, which was ravaged during the Muslim invasion.

Figure 1: An image of the main *maṇḍapa* of the Pāṭaṇādevī temple.

[3] This was undertaken by one of the editors, K. Ramasubramanian, along with three of his enthusiastic research scholars, K. Mahesh, Aditya Kolachana and Devaraja Adiga, between June 4-5, 2018. The credit of capturing nice photographs of the inscription as well as the other artefacts that are included here goes to the team members, and in particular to Devaraja Adiga.

On inquiring about the inscription, we were informed by the temple priest that currently the inscription is not available for public viewing due to renovation activity of the temple premises that has been undertaken by the ASI, and that this stone tablet has been safely kept along with other sculptures in a corner of the temple.

Figure 2: A section of the ravaged portion of the Pāṭaṇādevī temple.

However, since the trip was undertaken with the due permission of the director general of the Archaeological Survey of India, we were able to gain access to the inscription and examine first hand its current status. Here it may be added that it was due to the enthusiastic assistance and support offered by the priest, Sri Kishore Balakrishna Joshi,[4] who was on duty that day, that we were able to gain access to the inscription, as it was kept in a location which is inaccessible to the public.

As may be noted from the (full) image of the inscription given in Figure 3, a small portion of the top of the stone-tablet has been damaged. We were informed by the priest there, whose family has been associated with the temple for generations, that this inscription was moved from its original setting in the *maṭha* (which no longer exists) to the temple, long ago. The damage Kielhorn noted on a small portion of the top of the stone tablet was no doubt caused by this move. In addition, the surface erosion of this inscription

[4] Joshi ji was kind enough to introduce us to his father Sri Balakrishna Padmakar Joshi, whom we met at his residence in Chalisgoan and gather more information about the temple and its history.

(see Figure 4), which has rendered it almost illegible, may be related to its subsequent exposure to the elements.

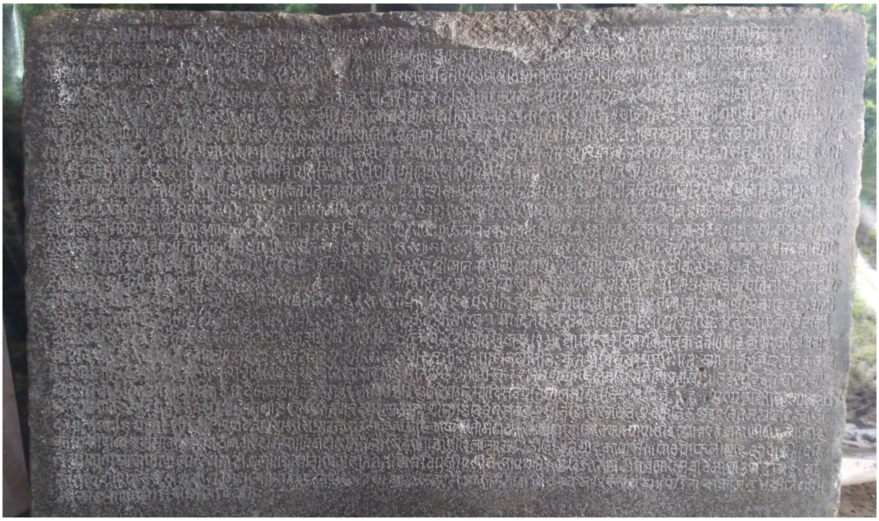

Figure 3: An image of the Caṅgadeva's inscription at the Pāṭaṇādevī temple.

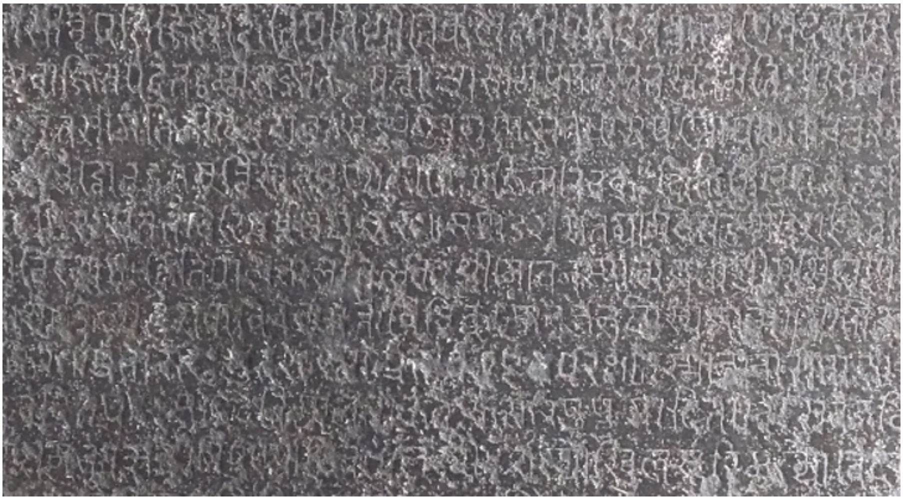

Figure 4: A magnified portion of a central part of inscription in Figure 3.

Based on living memory over generations, the priest was able to supply some details testifying to the significance of the temple and the site in which it is located. However, the knowledge of the precise location of the original *maṭha* has long since been lost. Though the temple seems to have been ravaged during the Mughal invasion, remnants of which are evident all over the temple (see Figure 5), the premises have been somewhat restored and people visit the temple in large numbers, during the festive seasons[5] to offer worship as well as perform certain special rituals.

Figure 5: The mutilated sculptures of the temple.

The priest also said that there were currently serious efforts by the ASI to further restore the temple and its surroundings to its original glory and celebrate the heritage of India's most famous mathematician. This was evident by fresh supplies of granite blocks, spread near the temple premises, that have been brought to the site for construction work. He further added that as soon as this is completed, the inscription will be once again available to be viewed and appreciated by the public, as a protected and celebrated heritage monument.

[5] According to the priest, the number of people who visit the temple would even cross 10,000 per day during special occasions.

3 Contents of the inscription

The inscription begins with a benediction, invoking the protection of the seven planetary deities. Then follow three ornate verses extolling Bhāskara's erudition in various branches of learning.

Verses 5–8 are devoted to the praise of the ruling prince, Siṃghaṇa Yādava of Devagiri (r. 1209–1247), his father Jaitrapāla (also known as Jaitrasiṃha or Jaitugi I, r. 1191–1209) and his grandfather Bhillama (r. 1187–1191) [Fle1882]. In verses 9–16 are described the chieftains of the Nikumbha dynasty, who, as the feudatories (*bhṛtya*) of the Yādava over lords of Devagari, ruled Khandesh, with their capital at Pāṭaṇ. These are Kṛṣṇarāja, his son Indrarāja, his son Govana and his son Soïdeva [Büh1879, pp. 39–42]. When the inscription was set up, Soïdeva was not alive any more and his younger brother Hemāḍideva had become the ruler (*soïdeve divaṃ yāte śāsti tasyānusambhavaḥ … hemāḍidevaḥ*, vv. 15–16).

Thereafter, the ancestors of Bhāskarācārya belonging to the *Śāṇḍilya-gotra* are described in verses 17–19. These are Trivikrama, his son Bhāskarabhaṭṭa who received the title *Vidyāpati* from Bhoja Paramāra of Dhārā (r. ca. 1010–1055), his son Govinda, his son Prabhākara, his son Manoratha and his son Maheśvara, who was the father of Bhāskarācārya.

Verses 20–22 describe Bhāskarācārya, his son Lakṣmīdhara, who was the chief pandit (*vibhudhāgraṇīh*) of the Yādava king Jaitrapāla, and his son Caṅgadeva, who became the chief astrologer of Jaitrapāla's son Siṃghaṇa (*siṃghaṇa-cakravarti-daivajña-varya*).

4 Anantadeva's inscription of 1222

Anantadeva, the grandson of Bhāskarācārya's brother Śrīdhara, founded a temple of goddess Dvārajā-devī (also known as Sārajā-devī) at Bahaḷ (20°36′ N; 75°9′ E). To record this event, his younger brother Maheśvara set up an inscription, which is dated Śaka 1144, Jovian year (southern style) *Citrabhānu*, *Caitra śukla* 1 (≡ Tuesday, 15 March 1222) [Kie1894, pp. 110–113].[6] This inscription gives some additional information about the family, in particular about the authorship of Maheśvara, the father of Bhāskarācārya and Śrīdhara.

An image of the dilapidated structure of the temple in which the Anantadeva inscription can be found in its original setting is shown in Figure 6. This temple, situated in a small cliff-top, is unfortunately in a neglected and

[6] See also [CESS1970, p. 41], s.v. Anantadeva, where a significant part of the inscription is reproduced.

ruined state. Since even the idols of the gods and goddesses do not exist in the sanctum sanctorum, which itself is open to the sky, the place is totally deserted. But fortunately, the structure above the spot in which the inscription is set is still in tact, and hence the inscription is not exposed to the elements and vagaries of nature. Therefore, unlike Caṅgadeva's inscription at the Pāṭaṇādevī temple, which has become badly eroded, this inscription is still well preserved and the text therein is perfectly readable. An image of the inscription is shown in Figure 7.

Figure 6: The dilapidated temple founded by Anantadeva at Bahal.

Combining the information of these two inscriptions, a genealogical tree can be drawn up as depicted in Figure 8 [Pin1981, p. 124].

5 Bhāskarācārya's ancestors and descendants

5.1 Trivikrama

Caṅgadeva's inscription mentions six ancestors of Bhāskarācārya and describes the sixth ancestor Trivikrama as *kavi-cakravartin* (v. 17). There are two other persons who bore the same name and with whom the present Trivikrama is sometimes identified; but there is no basis for such identification.

One of these is Trivikramabhaṭṭa, son of Nemāditya, who drafted the text of two land grants issued by the Rāṣṭrakūṭa king Indrarāja III [DrB1907,

Figure 7: An image of Anantadeva's inscription at Bahal.

pp. 24–41]. The last verse in both these grants, engraved on copper plates, reads thus:

> श्रीत्रिविक्रमभट्टेन नेमादित्यस्य सूनुना ।
> कृता शस्ता प्रशस्तेऽयं इन्द्रराजाङ्घ्रिसेविना ॥
>
> *śrī trivikramabhaṭṭena nemādityasya sūnunā* |
> *kṛtā śastā praśaste'yam indrarājāṅghrisevinā* ||
>
> This praiseworthy panegyric was composed by the illustrious Trivikramabhaṭṭa, son of Nemāditya (and) serving the feet of Indrarāja.
>
> (Translation by D. R. Bhandarkar)

Since these land grants were issued in 915, the Trivikrama who drafted these would be about half a century older than the present Trivikrama whose son received the title *Vidyāpati* from Bhoja Paramāra who reigned Malwa from ca. 1010–1055.

The other one is Trivikrama, of Śāṇḍilya-gotra, and son of Devāditya, who composed the *Nalacampū*, which is also known as *Damayantī-kathā* [NaCa1978, pp. 21–26]. Kielhorn identifies him with Bhāskarācārya's ancestor Trivikrama.[7] It is tempting to think that Bhāskarācārya inherited his fondness for śleṣa from the author of the *Nalacampū* which abounds in this kind of word-play. But there is no firm basis for this identification either, except

[7] There can be hardly any doubt that the *kavicakravartin* Trivikrama, with whom the list opens, is the *mahākavi* Trivikramabhaṭṭa, the author of the *Damayantī-kathā*, who, in the introduction of his work, describes himself as the son of Nemāditya (or Devāditya) and grandson of Śrīdhara, of the Śāṇḍilya *vaṁśa* [Kie1892, p. 340].

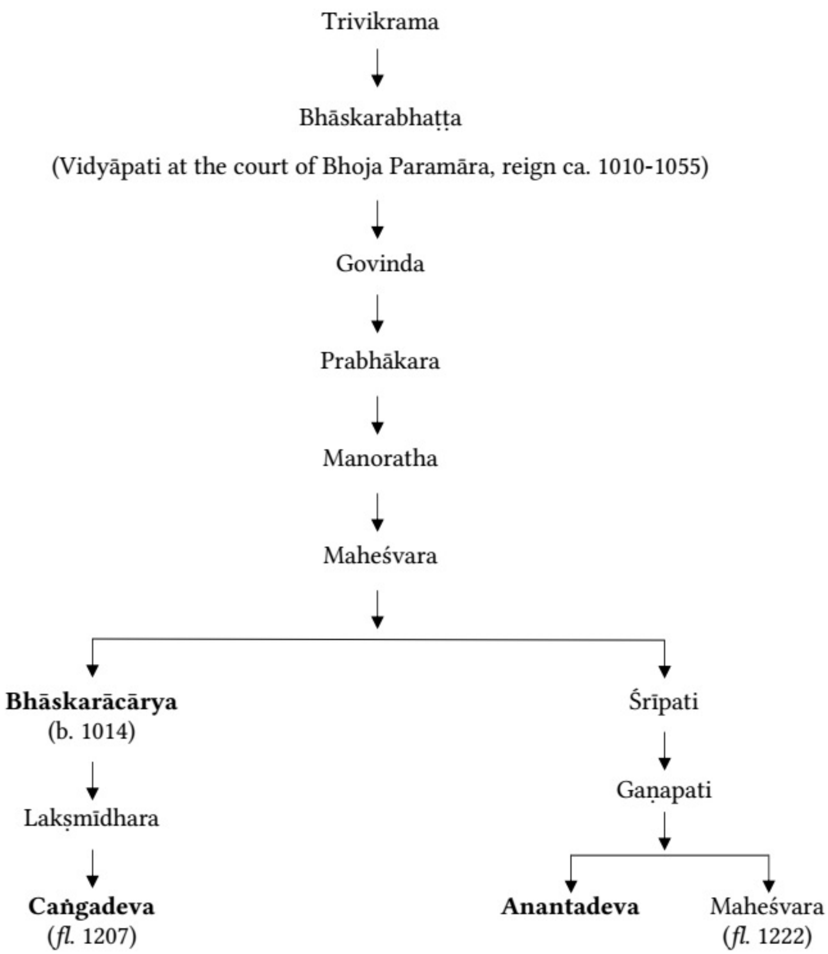

Figure 8: Genealogical tree of Bhāskarācārya.

that the author of the *Nalacampū* belongs to the same *Śāṇḍilya-gotra* like Bhāskarācārya. On the other hand, if we assume that the name Devāditya is a misreading for Nemāditya in the available manuscript corpus of the *Nalacampū*, its author could be identical with Trivikrama, the author of the land grants.

5.2 Bhāskarabhaṭṭa, the Vidyāpati at the court of Bhoja Paramāra

Caṅgadeva's inscription states that Trivikrama's son Bhāskarabhaṭṭa was given the title *Vidyāpati* by the king Bhoja (v. 17) of the Paramāra dynasty. Bhoja is renowned as a great patron of letters and himself an author of many works of diverse nature. To be made the *Vidyāpati* at his court must indeed have been a great honour. In modern parlance, *Vidyāpati* would correspond to the 'President of the Royal Academy of Letters', but it is not known what his duties and privileges at the Bhoja's court were. Bhāskarabhaṭṭa, or *Vidyāpati*, is not mentioned in any documents or legends related to Bhoja.

Bilhaṇa, the poet from Kashmir, also held the title of *Vidyāpati* at the court of Vikramāditya VI Cālukya (1076–1126) and this title conferred on him certain privileges like riding on a mighty elephant under a blue parasol, as he states in his *Vikramāṅkadevacarita* [ViCa1875, 18.101ab]:

> नीलच्छत्रोन्मदगजघटापात्रमुत्त्रस्तचोलात्
> चालुक्येन्द्रादलभत कृती योऽत्र विद्यापतित्वम् ।
>
> *nīlacchatronmadagajaghaṭāpātramuttrastacolāt*
> *cālukyendrādalabhata kṛtī yo'tra vidyāpatitvam* |
>
> There the lucky poet received from the Chālukya king, the terror of the Cholas, the dignity of Chief Paṇḍit, distinguished by the grant of a blue parasol and a mast elephant. (Translation by Bühler)

A similar title of *Vidyādhikārin* was enjoyed by the noted Telugu poet Śrīnātha (1365–1450) at the court of the Reddi kings who drafted some of their inscriptions [Ram1911, pp. 313–326]. In one of these dated Friday 21 February 1419, the Sanskrit part ends as follows:

> विद्याधिकारी श्रीनाथो वीरश्री-वेमभूपतेः ।
> [आ]करोदाकरो वाचं निर्मलं धर्मशासनम् ॥
>
> *vidyādhikārī śrīnātho vīraśrī-vemabhūpateḥ* |
> [*a*]*karodākaro vācaṃ nirmalaṃ dharmaśāsanam* ||
>
> Śrīnātha, the *Vidyādhikārin*, composed the text of this religious endowment (*dharma-śāsana*) of the illustrious warrior king Vema, the text which is a mine of faultless words.

The second inscription dated Wednesday 15 January 1416 concludes with the phrase '*śrīnāthakṛti*' (Śrīnātha's work). In the third inscription dated Tuesday 31 January 1413, there is no mention of the author of the inscription. Therefore, some historians concluded that the position of *Vidyādhikārin* at the royal court involved the drafting of the texts of the royal inscriptions.

But there is no evidence that Bhāskarabhaṭṭa composed the text of any inscriptions issued by king Bhoja. Pratipal Bhatia, who made a thorough study of all the inscriptions of the Paramāra dynasty, does not mention any inscription authored by Bhāskarabhaṭṭa [Bha1967]. Nor there is any evidence that Bilhaṇa composed the text of any inscriptions of Vikramāditya IV of Kalyan.

5.3 Maheśvara

About his own father Maheśvara, Bhāskarācārya states that this treasure house of boundless erudition (*niḥśeṣavidyānidhi*) and the crest-jewel of astrologers (*daivajña-cūḍāmaṇi*) resided in Vijjalaviḍa on the slopes of the Sahyādri range [SiŚi1952, *Praśnādhyāya*, v. 61, p. 300].

Caṅgadeva's inscription (v. 19) merely states that Maheśvara was a great poet (*kavīśvara*), but Anantadeva's inscription (vv. 3–4) enumerates four works composed by him, namely

1. *Karaṇaśekhara*,
2. *Pratiṣṭhāvidhi* or *Pratiṣṭhāvidhi-dīpaka*,
3. a *ṭīkā* on Varāhamihira's *Laghujātaka* and
4. a book on prognostication (*phala-grantha*).

Of these, the last two are extant; the *Laghujātaka-ṭīkā* is available in about ten manuscripts and the book on prognostication, entitled *Vṛtta-śataka* in 105 verses, is extant in several manuscripts [CESS1981, pp. 397–399].

5.4 Lakṣmīdhara

Bhāskarācārya's son Lakṣmīdhara is described in verses 22–23. Because Lakṣmīdhara was an expert in all branches of learning (*sakalaśāstrārthadakṣa*), the Yādava king Jaitrapāla (r. 1191–1209) took him away to his capital Devagiri and made him there the head of the learned men at his court (*vibudhāgraṇī*).

5.5 Caṅgadeva

Bhāskarācārya's grandson Caṅgadeva was said to be the chief astrologer of king Siṃghaṇa, who ruled the kingdom of Devagiri from 1209 to 1247 (*siṃghaṇa-cakravarti-daivajña-varya*). He was also the guru of the local chieftain Soïdeva of the Nikumbha dynasty.

5.6 Anantadeva

It will not be out of place to mention the available details from the Bahal inscription about Anantadeva, the grandson of Bhāskarācārya's brother Śrīdhara. According to this inscription (v. 7), Anantadeva wrote commentaries on the *Chandaścityuttara*, the twentieth chapter of the *Brāhmasphutasiddhānta* of Brahmagupta, and on the *Bṛhajjātaka* of Varāhamihira [Kie1894, p. 112].

> रम्यं ब्रह्मविनिर्मितं व्यवृणुतच्छन्दश्चितेरुत्तरम् ।
> होरां च प्रवरां वराहमिहिराचार्यप्रणीतं पृथुम् ॥
> *ramyaṃ brahmavinirmitaṃ vyavṛṇutacchandaściteruttaram |*
> *horāṃ ca pravarāṃ varāhamihirācāryapraṇītaṃ pṛthum ||*
>
> [He] beautifully explained the *chandaścityuttara* composed by Brahma [Gupta], as well as the highly acclaimed and elaborate *Horā* [*śāstra*] written by Varāhamihira.

However [CESS, Series A, vol. 1 (1970), p. 41] does not record any manuscripts of these two works. He was the chief astrologer (*daivajñagaṇāgraṇī*) of the Yādava king Siṃghaṇa (1209–1247), and built a temple of Dvārajā-devī at Bahal. As mentioned above, the construction of the temple was recorded in an inscription, which was drafted by Anantadeva's younger brother Maheśvara and engraved by the Nāgara Brahmin Pandit Gaṅgādhara, on 15 March 1222. The temple must have been built earlier.

6 Royal patronage

Some of these ancestors and descendants of Bhāskarācārya enjoyed royal patronage. Bhāskarācārya's fifth ancestor and namesake held the title of *Vidyāpati* at the court of Bhoja Paramāra in the first half of the eleventh century. Bhāskarācārya's son Lakṣmīdhara was made the chief of the learned persons (*vibudhāgraṇī*) by the Yādava king Jaitrapāla (r. 1191–1209) at his court

at Devagiri. The grandson Caṅgadeva was the chief astrologer of Siṃghaṇa (r. 1209–1247) at Devagiri as well as the guru of the Nikumbha chief Soïdeva.

Anantadeva from the collateral family was also the chief astrologer at the court of Siṃghaṇa, probably after the death of Caṅgadeva. There appears to have been some sort of rivalry between the cousins Caṅgadeva and Anantadeva, for neither Caṅgadeva's inscription makes any reference to his grandfather Bhāskarācārya's brother Śrīdhara and his descendants, nor does Anantadeva's mention his grandfather Śrīdhara's more famous brother Bhāskarācārya and his descendants!

Returning to the subject of royal patronage, unlike some of his ancestors and some of his descendants, Bhāskara does not appear to have sought royal patronage. If he was patronised by any king, Caṅgadeva would have mentioned it in his inscription. Bhāskara himself does not praise any contemporary king in any of his works. He probably remained at Vijjalavida or Bijjalbid throughout his long life and engaged himself in writing and teaching, although no names of his direct students have come down to us. This town does not exist any more, but most probably it lay close to the temple of Bhavānī where Caṅgadeva's inscription is located.

7 The *maṭha* founded by Caṅgadeva

The inscription states (vv. 23–25) that Caṅgadeva founded an institution (*maṭha*) for the propagation of the works composed by the illustrious Bhāskarācārya [Kie1892, v. 23]:

> श्रीभास्कराचार्यनिबद्धशास्त्रविस्तारहेतोः कुरुते मठं यः।
>
> *śrībhāskarācāryanibaddhaśāstravistārahetoḥ kurute maṭhaṃ yaḥ* |
>
> Who, to spread the doctrines of promulgated by the illustrious Bhāskarācārya, he founded a *maṭha*. (Translation by Kielhorn)

This institution was primarily meant to promote studies on the *Siddhāntaśiromaṇi* and other works composed by Bhāskarācārya and also the works by the other members of the family. In short, it was supposed to foster the tradition of scholarship and carry it forward in a manner that would have been cherished by Bhāskara. Among the members of the family, we know only of the works of Bhāskara's father Maheśvara, but not the works by other members if there are any. For the purpose of study, some of the works may have been copied at this *maṭha*, but no such manuscript copies have come down to us.

7.1 Foundation of the maṭha

When Caṅgadeva wished to establish a *maṭha*, the Nikumbha prince Soïdeva gifted a piece of land for it, on the occasion of a lunar eclipse, as stated in the inscription, in a prose passage thus [Kie1892, p. 346]:

> स्वस्ति श्रीशाके ११२८ प्रभवसंवत्सरे श्रावणमासे पौर्णमास्यां चन्द्रग्रहणसमये श्रीसोइदेवेन सर्वजनसन्निधौ हस्तोदकपूर्वकं निजगुरुमठाय स्थानं दत्तम् ।
>
> *svasti śrīśāke 1128 prabhavasaṃvatsare śrāvaṇamāse paurṇamāsyāṃ candragrahaṇasamaye śrīsoïdevena sarvajanasannidhau hastodakapūrvakaṃ nijagurumaṭhāya sthānaṃ dattam |*
>
> May it be well! In *Śaka* 1128, in the year Prabhava, on the full-moon day in the month of Śrāvaṇa, at the time of an eclipse of the moon, the illustrious Soïdeva, in the presence of all the people, granted land to the *maṭha* founded by his preceptor [confirming the gift] (see p. 22) by [pouring out] water from his hand.[8]

About the date on which the gift was made, Kielhorn observes:

> Soïdeva's grant … is dated in the *Śaka* year 1128, in the [Jovian] year [southern style] Prabhava, on the full-moon day of the month *Śrāvaṇa*, at the time of an eclipse of the moon. The date itself shows that there must be some mistake in it; for *Prabhava* corresponds to *Śaka* 1129 expired and not to *Śaka* 1128 expired. And, besides if the grant has been really made in *Śaka* 1128 expired, the date would fall in AD 1206, and in that year there was no lunar eclipse at all. The year of the grant therefore was clearly *Śaka* 1129 expired, which was the Jovian year *Prabhava*: and calculating for that year, I find that *Śrāvaṇa-sudi* corresponds to the 9th August, AD 1207, when there was a lunar eclipse, at 12 hours 26 minutes Greenwich time, or, at Ujjain, 11 hours 29 minutes after the mean sunrise. The eclipse, a partial one, lasted 2 hours 40 minutes and would, therefore, have been just visible in Khāndeś.

7.2 Date of the inscription

The inscription itself was not set up at the time of the land grant on 9 August 1207, but later for the following reasons. On 9 August 1207, Soïdeva was still alive and made the land grant. But the inscription was made after Soïdeva's death and when his younger brother Hemāḍi was ruling, as the inscription clearly states (verse 15: *soïdeve divaṃ yāte śāsti tasyānusambhavaḥ*, (see p. 8). The inscription also states that Caṅgadeva, besides being the guru of Soïdeva, was also the chief astrologer at the court of the Yādava king Siṃghaṇa (verse 23: *siṃghaṇa-cakravarti-daivajñavaryo … caṅgadevaḥ*). This

---

[8] The formal act of making a religious gift (*dāna*) consists in pouring water upon one's right palm and letting the water flow down.

Singhana ascended the throne of Devagiri in *Śaka* 1131 [Fle1882, pp. 71,73]. Therefore the inscription was set up at the earliest in *Śaka* 1131 (AD 1209–10) if not later.

7.3 Endowments for the maṭha

The inscription states that the Nikumbha chieftain Soïdeva and his younger brother and successor Hemāḍi, and also other people gifted land and other things to the *maṭha*. Moreover, certain kinds of taxes were levied for the regular maintenance of the *maṭha*. The details of these levies are written in a contemporary vernacular so that people who have to pay these taxes could know what their obligations were.

The sales tax on the goods sold in the market town, which is due to the king, is donated by the king to the *maṭha*. Likewise, the monies usually paid by the sellers to Brahmins, are to be donated by the latter to the *maṭha*. Also some part of the tax levied on pack-horses and bullocks would now go to the *maṭha*.

There were also levies in kind. Out of every hundred areca nuts (*pophalī*) purchased, the purchaser (*gāhaka*) was to donate five to the *maṭha*. Likewise a part of oil seeds brought to the oil mill would go to the *maṭha*, as also one ladle of oil from each oil mill should go to the *maṭha*, and so on.

7.4 Location of the maṭha

The inscription mentions that Soïdeva gave a piece of land for the *maṭha* in 1207, but does not mention the location or the area of the land. When the inscription was engraved a few years later, the stone slab on which it was engraved must have been set up somewhere in that *maṭha*. Probably when the *maṭha* ceased to exist, the stone slab with the inscription was moved to the temple of Bhavānī and was embedded in a wall. It is quite probable that the *maṭha* was close to the temple; otherwise the stone slab could not have been carried over a long distance. It is not known when the temple was built and by whom.

The temple is dedicated to Bhavānī or Caṇḍikā, but she is called Pāṭaṇā-devī in local parlance, in the sense of the presiding deity of the place 'Pāṭaṇ'. In the vicinity is a temple of Mahādeva, built by the Nikumbha prince Govana,

the father of Soïdeva and Hemāḍideva, in *Śaka* 1075. [9] Probably the Pāṭaṇā-devī temple was also built about the same time, as both the temples are said to follow the Hemadipanti style of architecture [Büh1879].

The *Gazetteer of the Bombay Presidency*, vol. XII, on Khandesh, carries the following description of the Pāṭaṇā-devī temple [GaZ1890, p. 482]:

> Half a mile from the village, towards the hill on the opposite or the east bank of the stream, is a temple of Devi. A flight of twenty-five steps, leading down to the stream, has on each side a lamp pillar, *dipmál*, one much older than the other. The building is a quadrangle, surrounded by stone and cement varandahs, *otás*, with a ruined roof and shrine. In the shrine are three cells in a line and a smaller cell facing the third cell. Two of the three main cells have *lings*, and two have images of goddesses and sacred bulls. The third with an image of Devi is the only one still worshipped. The small cell on the left has an image of Vishnu. In the vestibule are the representations of the Sheshasháyi, Devi and Lakṣmī Nārāyaṇa. The cells and the vestibules are built in the Hemádpanti style and the ground is paved. The building contains thirty-five pillars, some round and some four-cornered, and seven of them with new stone supports. The pillars and doors are to some extent ornamented. The ruined walls in places have been repaired with brick. The entire building is sixty-nine feet long, fort-five broad, and fourteen high. At an outer corner of the temple is a stone with a Sanskrit inscription.

Appendix: Caṅgadeva's inscription

उद्भटबुद्धिर्भाट्टे सांख्येऽसंख्यः स्वतन्त्रधीस्तन्त्रे ।
वेदेऽनवद्यविद्योऽनल्पः शिल्पादिषु कलासु ॥२॥ [आर्या]
स्वच्छन्दो यश्छन्दसि शास्त्रे वैशेषिके विशेषज्ञः ।
यः श्रीप्रभाकरगुरुः प्राभाकरदर्शने कविः काव्ये ॥३॥ [गीतिः]
बहुगुणगणितप्रभृतिस्कन्धत्रितये त्रिनेत्रसमः ।
विबुधाभिवन्दितपदो जयति श्रीभास्कराचार्यः ॥४॥ [उपगीतिः]

Triumphant is the illustrious Bhāskarācārya whose feet are revered by the wise, [he who was] eminently learned in Bhaṭṭa's doctrine, unique in the *Sāṁkhya*, an independent thinker in the *Tantra*, possessed of unblemished knowledge of the Veda, [and] great in mechanics and other arts; who laid down the law in metrics, was deeply versed in the *Vaiśeṣika* system, might have instructed the illustrious Prabhākara in his own doctrine, was in poetics [himself] a poet, [and] like unto the three-eyed [God Śiva] in the three branches [of the *Jyotiṣa*], the multifarious *Gaṇita* and the rest.[10]

[9] Midway between these two temples, the forest department of the State of Maharashtra set up a memorial for Bhāskarācārya with a statue of his in 2011.

[10] We preserve here Kielhorn's original translation, however we have provided an improved version in the epigram to Part I of this volume.

श्रीमद्यदुवंशाय स्वस्त्यस्तु समस्तवस्तुस्वहिताय ।
विश्वं यत्र त्रातुं जातो विष्णुः स्वतन्त्रस्तु ॥५॥ [आर्या]

May blessings rest on the illustrious race of Yadu with everything pertaining to it, [a race] in which Viṣṇu himself was born, to protect the trembling universe.

गर्जद्दुर्जरकुञ्जरोत्कटघटासंघट्टकण्ठीरवो
लाटोरस्ककपाटपाटनपटुः कर्णाटहृत्कण्टकः ।
श्रीमान् भिल्लमभूपतिः समभवद्भूपालचूडामणिः
त्रस्तार्त्तान्ध्रपुरन्ध्रिकान्तसुखहृच्छ्रीजैत्रपालोऽभवत् ॥६॥ [शार्दूलविक्रीडितम्]

[In this race] was born the illustrious prince Bhillama, a Lion to the furious combined arrays of the roaring Gurjara elephants, skilful in cleaving the broad breasts of the Lāṭas, [and] a thorn in the hearts of the Karṇāṭas. [And] here there was the crest-jewel of princes, the illustrious Jaitrapāla, who put an end to the pleasures of the beloved ones of the distressed ladies of Andhra.

लक्ष्मीकान्तलवः प्रतारितभवः श्रीजैत्रपालोद्भवः
सङ्ग्रामाङ्गणसञ्चितातिविभवः शास्ता भुवः सिंघणः ।
पृथ्वीशो मथुराधिपो रणमुखे काशीपतिः पातितो
येनासावपि यस्य भृत्यवटुना हम्मीरवीरो जितः ॥७॥ [शार्दूलविक्रीडितम्]

From the illustrious Jaitrapāla sprang, [in truth] a part of the beloved of Lakṣmī, Siṁghaṇa, who escaped [the ills of] this mundane existence(?); a ruler of the earth who acquired great might on the battle-fields, who in the van of the fight struck down the prince ruling over Mathurā [and] the lord Kāśī, [and] by whose young dependent even that valorous Hammīra was defeated.

अवततार पुरा पुरुषोत्तमो
यदुकुले जगतीहितहेतवे ।
जयति सोऽयमिमां सकलामिलां
अवति माम(पि)सिद्धमहीपतिः॥८॥ [द्रुतविलम्बितम्]

In former days Puruṣottama became incarnate for the good of the world in Yadu's family. He it is who here is conquering the whole earth [and] who protects me,– the ruler of the earth, the lord Siṁha.[11]

अथ भृत्यान्वयवर्णनम् ।

Now for the description of the family of the dependents

श्रीमद्भास्करवंशाय भव्यं भूयात् स भूपतिः ।
निकुम्भो यत्र संभूतो रामो यस्यान्वयेऽभवत् ॥९॥ [अनुष्टुभ्]

May fortune attend the illustrious solar race in which the prince Nikumbha was born, whose descendant was Rāma.

तत्रासीन्नृपतिर्जितक्षितिपतिर्घ्यातैकलक्ष्मीपतिः
दैवब्राह्मणचन्दने ततमतिः श्रीकृष्णराजाह्वयः।

[11] Here the translation seems to be based on the reading सिंहमहीपतिः.

शौर्यौदार्यविवेकविक्रमगुणैस्तुल्यो न येनापरः
प्रीत्या पाण्डवपुङ्गवार्जितपदं तद्धर्मराजेति यत् ॥१०॥ [शार्दूलविक्रीडितम्]

In this [race] was the illustrious prince, named Kṛṣṇarāja, who defeated the rulers of the earth, meditated solely on the lord of Lakṣmī, [and] bent his thoughts on revering gods and Brāhmaṇas. Since in the qualities of bravery, generosity, discrimination and prowess none else was his equal, he obtained that title which had been gained by the foremost of the Pāṇḍavas, [the title of] Dharmarāja.

प्राप्ताङ्गप्रभवस्ततस्ततमतिः प्राप्तप्रतापोन्नतिः
धीरो वैरिवधूविधूनितशिरा यः श्रीन्द्रराजाह्वयः ।
तस्यासीत्तनयः सतां सविनयः सामन्तसीमन्तिनी-
वैधव्यव्रतसद्गुरुर्गुरुगुणः सत्पुण्यपण्यापणः ॥११॥ [शार्दूलविक्रीडितम्]
चतुरस्तुरगारूढो रेवन्त इव गोवनः ।
सौन्दर्यदर्पः कन्दर्पो यं दृष्ट्वाऽनङ्गतां गतः ॥१२॥ [अनुष्टुभ्]

From him sprang a son, the illustrious Indrarāja, of far-reaching intelligence [and] endowed with eminent prowess; a hero, whose head was fanned by the wives of [hostile] feudatories the vow of widowhood, endowed with sterling qualities [and] a store-house of religious merit, Govana, skilful as a rider of horses like Revanta, at whose sight the god of love, proud as he was of his beauty, left the body.

श्रीगोवना[12] द्रत्नसिंधः उद्भूतमूर्त्तिस्ततपुण्यकीर्त्तिः ।
जितारिचक्रः क्षितिपालशक्रः श्रीसोइदेवः स्तुतवासुदेवः ॥१३॥ [उपजातिः]

From the illustrious Govana, an ocean, as it were containing countless jewels, sprang, a very Indra among the rulers of the earth, the illustrious Soïdeva, who spread the fame of his religious merit, conquered the hosts of enemies, [and] adored Vāsudeva.

शरणागतवज्रपञ्जरः परनारीषु सदा सहोदरः ।
व्रतसत्यपथे युधिष्ठिरः सततं वैरिवधूभयज्वरः ॥१४॥ [वैतालीयम्]

A cage of adamant to [shelter] those who sought his protection, always a brother to other's wives, in keeping the vow of truth a very Yudhiṣṭhara, [and] ever a fever of terror to the enemies' wives.

स षोडशशतग्रामदेशं दुर्गपुरान्वितम् ।
सोइदेवे दिवं याते शास्ति तस्यानुसम्भवः ॥१५॥ [अनुष्टुभ्]
त्यागे सूर्यसुतोपमोऽर्जुनसमः शौर्ये निकुम्भान्वये
विख्यातः क्षितिपालभालतिलकः श्रीगोवनस्यात्मजः ।
श्रीमत्सिंघणदेववैरिकरटीकण्ठीरवो यत्करः
नन्द्यान्नन्दसुनन्दने ततमतिर्हेमाद्रिदेवश्चिरम् ॥१६॥ [शार्दूलविक्रीडितम्]

Since Soïdeva has gone to heaven, his younger brother rules here 'the country of the sixteen-hundred villages' with its forts and towns.

May he, Hemādrideva, the son of the illustrious Govana, whose thoughts are fixed on Nanda's son, long live happily,- he who in liberality resembles the son of the Sun, [and] who in bravery is like Arjuna; that famous frontal ornament of the

[12] श्रीगोवनात्सज्जनरत्नसिन्धोरिति पाठे न छन्दोभङ्गः ।

princes in Nikumbha's family, whose hand is a lion to the elephants of the enemies of the illustrious Siṁghaṇadeva.

शाण्डिल्यवंशे कविचक्रवर्त्ती त्रिविक्रमोऽभूत्तनयोऽस्य जातः ।
यो भोजराजेन कृताभिधानो विद्यापतिर्भास्कर भट्टनामा ॥१७॥ [उपजातिः]

In the Śāṇḍilya race was the king of poets Trivikrama. To him was born a son, named Bhāskarabhaṭṭa on whom king Bhoja conferred the title of *Vidyāpati.*

तस्माद्गोविन्दसर्वज्ञो जातो गोविन्दसन्निभः ।
प्रभाकरः सुतस्तस्मात् प्रभाकर इवापरः ॥१८॥ [अनुष्टुभ्]

From him was born Govinda, the omniscient, like unto the Govinda; [and] he had a son, a second Sun, as it were, Prabhākara.

तस्मान्मनोरथो जातः सतां पूर्णमनोरथः।
श्रीमान् महेश्वराचार्यः ततोऽजनि कवीश्वरः ॥१९॥ [अनुष्टुभ्]

From him was born Manoratha, who fulfilled the desires of the good; [and] from him, the illustrious Maheśvarācārya, the chief of poets.

तत्सूनुः कविवृन्दवन्दितपदः सद्वेदविद्यालता-
कन्दः कंसरिपुप्रसादितपदः सर्वज्ञ (विप्रासदः) ।
यच्छिष्यैः सह कोऽपि नो विवदितुं दक्षो विवादी क्वचित्,
श्रीमान् भास्करकोविदः समभवत् सत्कीर्त्तिपुण्यान्वितः ॥२०॥ [शार्दूलविक्रीडितम्]

His son was the illustrious Bhāskara, the learned, endowed with good fame and religious merit, the root [as it were] of the creeper- true knowledge of the Veda, [and] an omniscient seat of learning, whose feet were revered by crowds of poets, while his words were rendered perspicuous by the enemy of Kaṁsa, [and] with whose disciples no disputant anywhere was able to compete.

लक्ष्मीधराख्योऽखिलसूरिमुख्यः वेदार्थवित्तार्किकचक्रवर्त्ती ।
क्रतुक्रियाकाण्डविचारसारो विशारदो भास्करनन्दनोऽभूत् ॥२१॥ [उपजातिः]

Bhāskara's son was Lakṣmīdhara, the chief of all sages,[13] who knew the meaning of the Veda, (*and*) who was the king of logicians (*and*) conversant with the essence of discussion on the subject of sacrificial rites.

सर्वशास्त्रार्थदक्षोऽयमिति मत्वा पुरादतः ।
जैत्रपालेन यो नीतः कृतश्च विबुधाग्रणीः ॥२२॥ [अनुष्टुभ्]

Judging him to be well acquainted with the contents of all the *Śāstras*, Jaitrapāla took him away from this town and made him chief of the learned.

तस्मात् सुतः सिंघणचक्रवर्तिदैवज्ञवर्योऽजनि चङ्गदेवः ।
श्रीभास्कराचार्यनिबद्धशास्त्रविस्तारहेतोः कुरुते मठं यः ॥२३॥ [इन्द्रवज्रा]
भास्कररचितग्रन्थाः सिद्धान्तशिरोमणिप्रमुखाः।
तद्वंश्यकृताश्चान्ये व्याख्येया मन्मठे नियतम् ॥२४॥ [उपगीतिः]

[13] Editorial note: Instead of 'the chief of all sages', 'foremost among the poets' would be a better translation of the word '*sūrimukhya*'.

To him was born a son Caṅgadeva, [who became] chief astrologer of king Siṁghaṇa; who, to spread the doctrines promulgated by the illustrious Bhāskarācārya, he founded a college, [enjoining] that in [this] his college the *Siddhāntaśiromaṇi* and other works composed by Bhāskara, as well as other works by members of his family, shall be necessarily expounded.

श्रीसोइदेवेन मठाय दत्तं हेमादि (ना किञ्चिदिहापरैश्च) ।
भूम्यादि सर्वैः परिपालनीयं भविष्यभूपैर्बहुपुण्यवृद्ध्यै ॥२५॥ [उपजातिः]

The land and whatever else has been given here to the college by the illustrious Soïdeva, by Hemādi and by others, should be protected by future rulers for the great increase of [their] religious merit.

स्वस्ति श्रीशाके ११२८ प्रभवसंवत्सरे श्रीश्रावणे मासे पौर्णमास्यां चन्द्रग्रहणसमये श्रीसोइदेवेन सर्वजनसन्निधौ हस्तोदकपूर्वकं निजगुरुचितमठायाग्रस्थानं दत्तम् ।

May it be well! In *Śaka* 1128, in the year *Prabhava*, on the full-moon day in the month *Śrāvaṇa*, at the time of an eclipse of the moon, the illustrious Soïdeva, in the presence of all the people, granted land to the college founded by his preceptor [confirming the gift] by [pouring out] water from his hand.

इयां पाटणीं जें केणें उघटे तेहाचा अमि आउंजी राउला होता ग्राहका पासी तो मठा दीन्हला । ब्राह्मणां जें विक[ते] यापासीं ब्रह्मोत्तर तें ब्राह्मणीं दीन्हलें ॥ ग्राह कापासीं दामाचा वीसोवा आसूपाठी नग[र] दीन्हला ॥ तलदा इया बैला सिद्ध[वें] ॥ बाहीरिला आसूपाठी गिधवें ग्राहकापामीं ॥ पांच पोफली ग्राहकापासीं ॥ पहिले आघाणे आदाणाची लोटि मठा दीन्हली ॥ जेती घाणे वांहति तेतीयां प्रति पलो पलो तेला ॥ एध[जें] मविजे तें मढीचेन मापें मवावें मापाउ मढा अर्द्ध ॥ अर्द्ध मापहारी ॥ [रु] पांचे सूंक । तथा भूमिः ॥ चतुराघाटविशुद्ध [ओडु?] ग्राम ॥ अ ...बाले ...कामतामध्यें च वं[टा] ॥ एकल-[टा] ॥ पंडितां [च?] कामतु ॥ [ची] ते ग्रामीचा [ऊ?]रा ॥ धामो[जी] ची[अ] सोढि[आ] ॥

In this market-town, whatever is exposed [for sale] by anyone is subject to assessment. That which would fall to the palace from the purchasers has been given to the *maṭha*. The Brahmottara, which the Brāhmaṇas have from the sellers, has been given by the Brāhmaṇas. The city has given the pack-horse tax, the twentieth of *drachma* [taken] from the purchasers. A settlement has to be made for the stall-receipts for the bullocks. The external pack-horse tax should be taken from the purchasers. Five areca-nuts [? out of a hundered] to be taken from the purchasers. A roller of [oil] seeds of the first oil-mill [rolling] has been given to the *maṭha*. A ladle each from as many oil-mills as are running. Here what is measured, should be measured by the measure in the *maṭha*; half the quantity measured to the *maṭha* and half to the measure. [Also] the cess on clarified butter. Thus the land, the boundaries ...in Bālā's service-land ...paṇḍit's service-land.

[The last lines are partly effaced.]

The legend of *Līlāvatī*

Sreeramula Rajeswara Sarma*

1 Introduction

Unlike the writers on other branches of Sanskrit learning about whom we know nothing but their names, authors of works on *Jyotiḥśāstra*, from Āryabhaṭa onwards, generally mention the years of their birth or epochs that are closer to their own times. Bhāskarācārya states in his *Siddhāntaśiromaṇi* that he was born in *Śaka* 1036 (=1114 CE) and that he composed the *Siddhāntaśiromaṇi* at the age of 36, i.e., in 1150 [SiŚi1981, *Gola, Praśnādhyāya*, v. 58]. He also mentions there the name of his father and his own place of residence [ibid. vv. 61–62]. In his *Karaṇakutūhala*, he mentions the epoch which corresponds to 23 February 1183 and states that he was the son of Maheśvara of *Śāṇḍilya-gotra* [KaKu1991, ch. 1.2–3; ch. 10.4]. More important, Bhāskara is the only astronomer for the propagation of whose works a *maṭha* was established. Bhāskara's grandson Caṅgadeva, who set up the *maṭha*, recorded this fact in an inscription of 1207, where he also gives a detailed genealogy of his family [Kie1892, pp. 338–346].

Thus Bhāskara is not a mythical figure, but a historical personage whose date, provenance and authorship are firmly established. Even so, legends grew about him and are perpetuated like the legends of Archimedes' bathtub or Isaac Newton's apple. Legends are, of course, interesting from the viewpoint of cultural history, but they obscure the actual life and achievements of the person concerned. In the age of the internet, legends spread far more widely than the scientific achievements. It is one of the tasks of the history of science to dispel the legends and establish the true facts of the life and achievements of the scientists of the past ages.

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2 The legend

The most persistent legend about Bhāskara relates that he composed the *Līlāvatī* to console his widowed or unmarried daughter named *Līlāvatī*. This legend occurs for the first time in 1587 in the Persian translation of the *Līlāvatī* by Akbar's Poet Laureate, Abū'l Fayḍ Fayḍī. In the preface, Fayḍī narrates as follows:[1]

> *Līlāvatī* was the name of his daughter. From … the horoscope of her birth, it appeared that she would be childless and spend her life without a husband. Her father, after much meditation, chose a particular moment for her marriage that would … enable the daughter to have children and progeny. … When this moment approached, he had the daughter and the groom brought together and set up a water clock *(tās-i sācāt)* to determine the correct moment.
>
> Since what was proposed was not in accordance with what was fated, it chanced that the girl, out of curiosity …, went on looking at the bowl of the water clock and enjoyed the sight of water coming in through the hole below. Suddenly, a single pearl … got detached from the bride's clothes and fell into the bowl and … stopped the inflow of water …. Thus the moment they had been looking for had passed.
>
> Finally, the luckless father told the ill-starred daughter: 'I shall write a book titled after your name, which will long endure in the world, for a good name is like a second life for one and confers immortal life upon the seed.'

From this time onwards, this legend spread far and wide and is accepted today as a historical fact.

In this paper, we examine the veracity of this legend on the basis of internal evidence from Bhāskara's own works and external evidence from the commentaries on Bhāskara's works and contemporary social mores. It will be shown that there is no truth in the story and that it was a mere legend invented to explain the unusual title of the book. We shall also attempt to trace the origin of the legend and draw attention to parallel legends of this nature.

3 Against the norms of *Dharmaśāstra*

From the outset it is obvious that naming a book after an unmarried or widowed daughter would have violated the social mores of the period. Fayḍī narrates that Bhāskara's daughter *Līlāvatī* remained unmarried. He would not have known that keeping a daughter unmarried was considered a grave sin in *Dharmaśāstra*. In this connection, the *Yājñavalkya-smṛti* declares that

[1] For the complete preface, see the Appendix 1 of this paper.

'[A father,] who does not give away his daughter in marriage, commits the sin of killing the embryo, each time she has her monthly course.'[2] As a scrupulous follower of tradition, Bhāskara would not have gone against the injunction of *Dharmaśāstra.*

On the other hand, according to yet another popular version now current in India, *Līlāvatī* was married, but the husband died soon after and *Līlāvatī* became a widow. To console her Bhāskara named his book after her. This version too goes against the traditional value system in India which considers the widow inauspicious. Bhāskara certainly would not have named his book after a widowed daughter howsoever dear she might have been to him.

4 Internal evidence

More important than this is the internal evidence from Bhāskara's own work which also shows the improbability of the legend. It is often asserted that the book was written for the daughter *Līlāvatī*, because the word occurs in the first and the last verses of the book and because some examples (*uddeśaka, udāharaṇa*) are directly addressed to her by name.

Before examining these cases, it is necessary to understand Bhāskara's aim in writing his works and the style which he adopts for this purpose. In each of his books, he repeatedly states that his aim is to make the subject lucid so that it is easily comprehensible to the beginners (*bāla-līlā-avagamya*), but at the same time the work should have enough new material, new methods of presentation, new turns of expression, and rationales (*upapattis*) which would earn the appreciation of experts (*catura-prīti-prada*).

Moreover, Bhāskara is not just an eminent mathematician and astronomer, but an accomplished poet as well. In fact, he often refers to himself as *kavi* Bhāskara.[3] As *kavi* he infuses his works with poetic flavor[4] through *līlā* and *lālitya. Līlā* denotes, on the one hand, 'play, amusement, pastime, child's play, ease or facility in doing anything,' and on the other, 'grace, charm, beauty,

[2] [YāSm, 1.64]: *aprayacchan samāpnoti bhrūṇahatyām ṛtāv ṛtau* |
Similar statements can be found also in other texts on *Dharmaśāstra.*

[3] [SiŚi1981, *Gola*, *Praśnādhyāya*, v. 62]: *siddhāntagrathanaṃ kubuddhimathanaṃ cakre kavir bhāskaraḥ*; [KaKu1991, *Parvasambhavādhyāya*, v. 4]: *tatsūnuḥ karaṇaṃ kutūhalam idaṃ cakre kavir bhāskaraḥ.*

[4] It is generally thought that Bhāskara was the first mathematician to introduce poetry in his work, but even before Bhāskara, Mahāvīra also tried in his *Gaṇitasārasaṅgraha* to enliven the dry arithmetic with poetic flourishes. One of his examples can be treated as a minor *kāvya*. Cf. [Sar2004a, pp. 463–476].

elegance, loveliness' and so on; in short, anything that can be performed with ease and cheerfulness. And *lālitya* means 'elegance'. Besides *līlā* and *lālitya*, Bhāskara also employs various figures of speech, including paranomasia (*śleṣa*).

5 The first verse of the *Līlāvatī*

This should be kept in mind when reading the first verse of the *Līlāvatī* [Līlā1937, v. 1]. After worshipping Gaṇeśa in the first half, Bhāskara continues the second half as follows:

> पाटीं सद्गणितस्य वच्मि चतुरप्रीतिप्रदां प्रस्फुटां
> संक्षिप्ताक्षरकोमलामलपदैर्लालित्यलीलावतीम् ॥
>
> *pāṭīṃ sadgaṇitasya vacmi caturaprītipradāṃ prasphuṭāṃ*
> *saṃkṣiptākṣarakomalāmalapadairlālityalīlāvatīm* ||
>
> I shall expound [the book of] algorithms of good mathematics (*sadgaṇitasya pāṭīm*), in words concise, soft and faultless (*saṃkṣiptākṣara-komalāmala-padair*), the book which would delight experts (*catura-prīti-pradām*), which is clear and unambiguous (*prasphuṭām*), and endowed with playful elegance (*lālitya-līlāvatīm*).

The expression *lālitya-līlāvatī* at the end of the verse does not refer to Bhāskara's daughter *Līlāvatī* but to *pāṭī*, the subject of the book. Therefore *lālitya-līlāvatī* implies that this book is elegant and is easy and pleasurable to read.[5] In fact, no commentator saw here a reference to a daughter called *Līlāvatī*.

Not only in the *Līlāvatī*, but also in his others works, Bhāskara employs the terms *līlā* and *lālitya* (or *lalita*) repeatedly in order to emphasize that his works are elegant, pleasurable and easy to comprehend, besides being innovative [BīGa1930, p. 225]; [SiŚi1981, *Madhyamādhikāra, Kālamānādhyāya*, v. 3 and *Gola*, *Golapraśaṃsā*, v. 9].

6 *Līlāvatī* as the addressee

In two verses, the *udāharaṇas* are addressed to a person described as *Līlāvatī*. Verse 13 begins with the words, 'O young and intelligent *Līlāvatī* (*aye bāle*

[5] Bhāskara employs the expression *lālitya-līlāvatī* also in the invocation at the beginning of the *Golādhyāya* and explains it in his *Vāsanā-bhāṣya* as *mādhurya-guṇa-sampannā* (endowed with graceful style): [SiŚi1981, *Golapraśaṃsā*, v. 1, p. 326].

līlāvati matimati), if you are good in addition and subtraction,' and asks her to solve a problem of simple addition and subtraction. Verse 17 is addressed to the 'young *Līlāvatī*, whose eyes are tremulous like those of a young deer' (*bāle bālakuraṅgalolanayane līlāvati*) and poses a simple problem of multiplication and division.

There is no doubt that in these two verses the author appears to be addressing a young female person as *Līlāvatī*. It may be her name or an attribute meaning that the person is a charming woman. But if Bhāskara wrote the book for his daughter *Līlāvatī*, all the *udāharaṇas* in this book must be addressed to her. But only 7 *udāharaṇas* have feminine addresses,[6] whereas as many as 47 *udāharaṇas* contain masculine addresses.[7] Even some of the feminine addresses are inappropriate for a daughter: *kalyāṇini* would be highly inappropriate for a girl whose marriage (*kalyāṇa*) did not take place or ended in her widowhood; the address *kānte* is better suited for a beloved and not for a daughter.[8]

The term '*Līlāvatī*' occurs also in verse 202 (*vimalāṃ ced vetsi līlāvatīm*, 'if you know the flawless *Līlāvatī*'), but here it refers, as in verse 1, to the work of this name and definitely not to a supposed daughter.

7 *Līlāvatī* in the last verse

The last verse 272 provides the clinching evidence against the hypothesis of a daughter named *Līlāvatī*; it is a verse with teasing layers of double meaning (*śleṣa*):

येषां सुजातिगुणवर्गविभूषिताङ्गी
शुद्धाऽखिलव्यवहृतिः खलु कण्ठसक्ता ।

[6] Besides the address *līlāvati* (vv. 13, 17), there are *bāle* (vv. 50, 67, 69), *cañcalākṣi* (v. 50), *mṛgākṣi* (v. 55), *kalyāṇini* (v. 71) and *kānte* (v. 71).

[7] The largest number of addresses are masculine: *kovida, gaṇaka* (12 times), *gaṇitakovida, gāṇitika* (3 times), *nandana, mitra* (5 times), *vaṇik* (3 times), *vaṇigvara, vatsa* (2 times), *vayasya, vicakṣaṇa, vidvan* (2 times), *sakhe* (23 times), *sumate* (2 times).

[8] In the *BG* also there are some feminine addresses like *kalyāṇini, kānte, priye, mṛgākṣi*, but here also the masculine addresses preponderate: *kuṭṭakajña, komalāmalamate, gaṇaka* (9 times), *gaṇitajña, tāta, bījagaṇitajña, bījajña, bījavittama, bhāvitajña* (*bhāvita* is a section in the *Bījagaṇita*), *vaṇik, vatsa, vidvan, sakhe* (15 times), *sumate*. The following addresses occur in the [SiŚi1981, *Gola, Praśnādhyāya*]: *gaṇaka* (6 times), *golakṣetravicakṣaṇa, tāntrika, mitra, vidvan, sakhe*. For comparison, Śrīdhara's *Pāṭīgaṇita* contains the following addresses: *gaṇaka, gaṇakottama, vaṇik, vidvan* (5 times) and *sakhe* (7 times), cf. [PāGa1959].

लीलावतीह सरसोक्तिमुदाहरन्ती
तेषां सदैव सुखसंपदुपैति वृद्धिम् ॥

yeṣāṃ sujātiguṇavargavibhūṣitāṅgī
śuddhā'khilavyavahṛtiḥ khalu kaṇṭhasaktā |
līlāvatīha sarasoktim udāharantī
teṣāṃ sadaiva sukhasampadupaiti vṛddhim ||

Those who memorize (*kaṇṭha-saktā* = *kaṇṭhastha*) the [book] *Līlāvatī*, whose sections (*aṅga*) are adorned with excellent [rules for] the reduction of fractions to a common denominator (*su-jāti*), multiplication (*guṇa*), squaring (*varga*) [etc.], having entirely accurate [fundamental] procedures (*śuddha-akhila-vyavahṛti* = *vyavahāra*), illustrating (*udāharantī*) [the rules] with elegant (*sarasa*) examples (*ukti*), …their happiness (*sukha*) and wealth (*sampat*) increase for ever and ever in this world; [in the same manner as] those whose neck is clasped (*kaṇṭha-saktā*) (i.e. embraced) by a charming woman (*līlāvatī*), whose limbs (*aṅga*) are adorned with a multitude of good qualities (*guṇa-varga*) [indicating her] good birth (*sujāti*), perfect in all her acts of behavior (*śuddha-akhila-vyavahṛtiḥ*), whispering passionate words (*sarasoktim udāharantī*), …the intensity of their [erotic] pleasure (*sukha-sampat*) increases for ever and ever.

Surely Bhāskara would not have composed this verse with such an erotic undertone about his own daughter! No commentator sees here a reference to his daughter. They see here two levels of meaning, one pertaining to the book and another pertaining to a graceful woman.[9] Thus Bhāskara's own work contains no reference to a daughter called *Līlāvatī*, nor do any of the commentaries mention a woman or daughter called *Līlāvatī*. Then how did the story of *Līlāvatī* appear in Fayḍī's Persian translation of the *Līlāvatī*?

8 The probable source of the legend in Fayḍī's translation

The Mughal emperor Akbar desired that the Muslim intelligentsia be made familiar with the classics of Hindu thought so that they have a better interaction with Hindus; therefore he established a bureau of translation, where some Sanskrit texts like the *Mahābhārata, Rāmāyaṇa* and *Pañcatantra* were rendered into Persian [AIF1873, I, 103–104]; [Riz1975, ch. 6, pp. 202–222]. In this bureau, translation was not performed by a single scholar proficient in

[9] For example, Mahīdhara in his *Līlāvatīvivaraṇa* explains thus: *yeṣām iha loke līlāvatīgrantho nārī ca kaṇṭhasaktā teṣāṃ nityaṃ sukhasampad vṛddhiṃ prāpnoti* (those who know by heart the book *Līlāvatī* and those who are embraced by a woman, their happiness and wealth increase constantly); see also Gaṇeśa [Līlā1937, p. 285]: *athāṅganāśleṣeṇa svakṛtapāṭīstavanarūpaṃ granthasamāpty-alaṃkāraṃ siṃhoddhatayāha* (now he utters the 'ornament' at the conclusion of the book, by praising the *pāṭī* composed by him through [the metaphor] of the embrace by a woman).

both Sanskrit and Persian, but by teams of persons, some proficient in Sanskrit and others in Persian, in three stages. First, Hindu or Jaina scholars of Sanskrit prepared a paraphrase in contemporary Hindi of the Sanskrit text to be translated. In the second stage, this Hindi paraphrase was translated into Persian by one of the several Muslim courtiers. Finally, the Persian translation was polished and put into elegant prose and verse by an established Muslim scholar at the court [Hod1939, I, pp. 565–566]. What resulted in this process cannot be termed an exact translation but rather a Persian paraphrase.

Bhāskarācarya's *Līlāvatī* also was rendered in this manner into Persian by Fayḍī with the help of Sanskrit scholars from the Deccan [AIF1873, I, p. 105]. John Taylor says that the Persian version has many lacunae, and departs in some passages so far from the original as to 'induce the suspicion that Faizi contended himself with writing down the verbal explanation afforded by his assistant [Tay1816, p. 2].'

Fayḍī must have wondered why the book of dry mathematics was given the name '*Līlāvatī*', a name that was usually given to girls. Five centuries prior to this time, al-Bīrūnī was likewise intrigued about the nomenclature of Brahmagupta's *Khaṇḍakhādyaka*, which literally meant 'a preparation made with candied sugar.' His interlocutor, with tongue in cheek, told him an amusing but improbable story which the haughty al-Bīrūnī faithfully recorded in his book on India [AIB1910, I, p. 156].

Like al-Bīrūnī, Fayḍī also must have demanded from the informant: '*Līlāvatī*' is a girl's name. Why did Bhāskarācārya give his book such a name?' Instead of explaining that giving such feminine names to books is not uncommon in Sanskrit tradition [BrSū1997],[10] the informant appears to have narrated a story about Bhāskara's daughter *Līlāvatī* and how she missed the exact time of marriage because the hole in the water clock got blocked. Fayḍī's informant did not invent the story entirely, but adapted a story that was current at that time in the popular didactic literature of the Jainas.[11]

9 The Jaina legend

In this Jaina story, a brahmin named Vidyānanda saw that his daughter's horoscope indicated her husband's death in the sixth month after the marriage. But one day he found such an auspicious moment that, if a girl was

[10] He could have mentioned the *Bhāmatī* of Vācaspati Miśra on *Advaita Vedānta, Bhāsvatī* of Śatānanda on astronomy, *Kāśikā* of Vāmana and Jayāditya on grammar and so on.

[11] It was included in Hemavijayagaṇī's *Kathāratnākara* [KtRa1911]. See Appendix 2 (of this chapter) for the complete story.

married at that moment, she would never become a widow. He decided to marry his daughter to the wealthy son of a brahmin at that auspicious moment and set up a water clock to indicate the moment. But he failed to notice that a grain of rice got loose from his forehead mark, fell into the bowl, and blocked the hole at the bottom. Since the hole in the water clock was blocked, the time of the auspicious moment passed without anybody realizing it.

Nevertheless, Vidyānanda performed the daughter's marriage. After six months, the husband died from snake bite and the daughter became a widow. The story concludes with the moral that 'Fate's writing cannot be altered' (*vidhinā likhitam anyathā na syāt*).

10 Differences between the Jaina story and Fayḍī's version

In the Jaina story the brahmin astrologer is named Vidyānanda, but the daughter's name is not mentioned. Fayḍī's informant substituted the name of Vidyānanda with Bhaskarācārya and gave the daughter the name *Līlāvatī*. There are other differences. In the Jaina story, a grain of rice from Vidyānanda's forehead mark fell into the bowl of the water clock and blocked its narrow hole. In Fayḍī's narration the water clock was blocked by the pearl that fell down from the ceremonial garment of *Līlāvatī* who was curious to see the time-measuring device. But in Bhāskarācārya's milieu, marriage is a very solemn ritual and the young bride would be sitting demurely at the prescribed place and not wander around. As befitting the ritual, she would also be wearing freshly washed and dried simple clothes and not pearl-studded garments. Moreover, a corn of rice with a narrow pointed tip has a greater chance of blocking the hole in the water clock than a round pearl.

But the major difference is this. In the original story, Vidyānanda marries off the daughter even though the prescribed moment had passed, knowing fully well that his dear daughter would soon become a widow. Fayḍī appears to have thought that it would be too cruel and made *Līlāvatī* remain unmarried, unaware that for a brahmin not to marry off the daughter would be a greater sin than having a widowed daughter.

11 Parallel legends in Jaina didactic literature

Such stories were common in Jaina didactic literature; these were meant to emphasise the maxim that mere knowledge of *śāstras* does not equip one to fully comprehend the reality; one needs to have spiritual intuition that can be attained only by those who renounce the world. Particularly relevant is the

legend in which Varāhamihira and the Jaina saint Bhadrabāhu, both historical personalities but belonging to different periods, are brought together as brothers [PrCi1933, pp. 118–119]. In this legend, the lay person Varāhamihira with all his *śāstra-jñāna* could not predict the correct length of the life of his own son, but the Jaina monk Bhadrabāhu could make the correct prediction by virtue of his *śruta-jñāna*.[12]

The story occurs also in other collections like Rājaśekhara Sūri's *Prabhandhakośa* [Gra1993, pp. 1–4].

12 Spread of the *Līlāvatī* legend in the west and in India

Fayḍī's legend of *Līlāvatī* lay dormant for the next two hundred and odd years until the officers of the East India Company stumbled upon it and put it into English at the beginning of the eighteenth century. Edward Strachey, who read the Persian translations of both the *Līlāvatī* and *Bījagaṇita* and who was to publish in 1813 an English translation of the Persian rendering of the *Bījagaṇita* by Ata Allah Rushdi [Str1813], communicated a summary of the *Līlāvatī* legend as narrated by Fayḍī to Charles Hutton, the 'Late Professor of Mathematics in the Royal Military Academy, Woolwich,' who published it in 1812 in the second volume of his authoritative *Tracts on Mathematical and Philosophical Subjects*.[13]

Soon after, John Taylor, in the introduction to his English translation of the *Līlāvatī* of 1816, cautioned that the legend narrated by Fayḍī 'has not been confirmed to me by any native of this country, nor have I observed it mentioned by any Hindu author [Tay1816, p. 3].'

Even so, the legend began to spread in the west and in India and was repeated innumerable times in almost all histories of mathematics, sometimes

[12] *Śruta-jñāna* is defined thus in the *Sarvadarśanasaṅgraha: jñānāvaraṇakṣayopaśame sati matijanitaṃ spaṣṭaṃ jñānaṃ śrutam* (*Śruta* is the clear knowledge produced by *mati*, when [all] obstructions of knowledge are destroyed and tranquility is achieved). Cf. Mādhavācārya, [SDS1962, pp. 52–53].

[13] In this volume, Tract XXXIII (pp. 143–305) is devoted to 'History of Algebra of all Nations', including a '[History] Of the Indian Algebra' (pp. 151–179) with extracts from the 'Leelawuttee' and 'Beej Gunnit', which were communicated to Hutton by the servants of the East India Company like Samuel Davis and Charles Wilkins. At the end of this account, Hutton introduces the *Līlāvatī* legend [Hut1812, pp. 177–178] with the following words: 'Since the foregoing account of the Indian Algebra was printed, I have been favoured by Mr. Strachey with the following translation of the Persian translator's preface to the *Līlāvatī;* which being at once very curious, and containing some useful particulars, is given below as a postscript to that account.'

reproducing Hutton's account verbatim and sometimes with further elaborations.[14] The only honourable exceptions appear to be Henry Thomas Colebrooke and Sankara Balakrishna Dikshit, who did not take this legend seriously and did not make any reference to it in their respective works [Col1817] [Dik1981, pp. 114–123].

Already in 1892, Sudhakara Dvivedi narrated two versions of the legend in his *Gaṇakataraṅgiṇī*: Bhāskara composed the *Līlāvatī* to console his daughter who became a child widow or to console his childless wife [GaTa1892, pp. 34–42].

According to M. D. Pandit four different versions are known in India:

(i) *Līlāvatī* was Bhāskara's daughter, who became a widow;

(ii) *Līlāvatī* remained unmarried;

(iii) *Līlāvatī* was Bhāskara's wife who did not have any children;

(iv) *Līlāvatī* was the daughter of Bhāskara's teacher who fell in love with Bhāskara, but Bhāskara could not reciprocate the love because, being the daughter of his teacher, she was like his own sister.

To console her, Bhāskara named his book after her. Very wisely Pandit rejects all the versions as untrue [Pan1922, pp. 10–11].

With the onset of the Internet, the *Līlāvatī* legend is better known today than Bhāskara's scientific achievements. In fact, *Līlāvatī* is hailed as a great lady mathematician of earlier times and as the role model for women scientists of today. A few examples should suffice.

In 2001, the National Science Centre, New Delhi, organised an exhibition on '*Līlāvatī:* Indian Women in Science'; the Director General of the Centre explained that the 'exhibition had been named so to signify the spirit of learning that *Līlāvatī* had shown hundreds of years ago. She symbolises the tradition of women's education in India.'[15] A recent publication on the women scientists of India carries the title *Līlāvatī's Daughters* [GR2008]. The well-known company Infosys of Bangalore has instituted recently the '*Līlāvatī* Prize' of

[14] An extreme case of a highly imaginative version of the legend is narrated by Edna Kramer who portrays Bhāskara as rebellious youth fighting against the restrictions of caste. Her account begins thus: When as a boy Bhāskara was not absorbed in the study of mathematics, he would brood on the nature of the society about him. The adamant wall of custom had always been a source of pain to him. He chafed under the system that forbade him to share his scientific knowledge with youths of lower caste, or to seek companionship outside brahmin ranks. Bhāskara had feared to confide his unorthodox views to others, lest he be outlawed [Kra1955, p. 1].

[15] *The Hindu*, New Delhi edition, 15 September 2001.

one million Rupees for the popularization of mathematics, to be awarded by the International Conference of Mathematicians.[16]

However, as we saw, there is no truth in this legend. It neither proves Bhāskarācārya's great mastery of predictive astrology, nor does it prove women's achievements in mathematics in ancient times in India. It merely distracts us from the more important scientific achievements of Bhāskarācārya.

Appendix 1

Fayḍī's preface to his translation of the *Līlāvatī*[17]

In accordance with the exalted order [of] His Majesty, [Fayḍī] submits a translation from the Hindī language (i.e., Sanskrit) into Persian of the *Līlāvatī*, which is reputed, among the unique works of arithmetic and mensuration, for its fluency and elegance of its style. Before coming to the beginning of the translation (lit. object), he (i.e., the translator) wishes to say that the author of the present work was Bhaskar Acharaj (*Ācārya*), the famous scholar who was unsurpassed in his time in the knowledge of mathematics. His native place was the city of Bidar in the region of Deccan. Although the date of the composition of the present work is not known, yet he has written another book on the methods of drawing up almanacs (*taqwīm*) and important secrets of astronomy (*tanjīm*) called Karanakutūhay (*Karaṇa-kutūhala*). There he has given the date of writing as 1105 of the era of Salibahan, from which till this year, which is 32 of the *Ilāhī* era and 995 by the lunar (*Hijrī*) calendar, 373 years have passed.[18]

As for the reason behind the writing of the book, it has been heard that *Līlāvatī* was the name of his daughter. From the indications of the horoscope of

[16] *Leelawati* Prize, *http://www.mathunion.org/general/prizes/leelavati/details/*. Of course, the prize is not meant exclusively for women; the two recipients so far are men. The first recipient in 2011 was the British author Simon Lehna Singh and the second recipient in 2014 Adrián Paenza from Argentina.

[17] I am highly grateful to Professor Irfan Habib who kindly translated the preface for me on the basis of a manuscript from Alwar (3644) and [Fai1827].

[18] The year of the composition of the *Karaṇakutūhala* is given correctly as 1105 Salibahan (i.e. *Śaka*) which translates to 1183 CE. As regards to the date of Fayḍī's preface, the *Ilāhī* year 32 lasted from 21 March 1587 to 20 March 1588 and the *Hijrī* year 995 from 12 Dec 1586 to 1 Dec 1587. Therefore, Fayḍī must have drafted the preface between 21 March 1587 and 1 Dec 1587. Then the difference between the composition of the *Karaṇakutūhala* and Fayḍī's preface should be 1587 – 1183 = 404 years and not 373 years as Fayḍī calculated.

her birth, it appeared that she would be childless and spend her life without a husband. Her father, after much meditation, chose a particular moment for her marriage that would be of a firm foundation of union and would enable the daughter to have children and progeny. It is said that when this moment approached, he had the daughter and the groom brought near each other and put the hour-bowl (*tās-i sā'āt*) in the full water vessel, having an astronomer present who would be able to determine the correct moment. It was agreed that, when the bowl settled down in the water, they would tie the wedding-knot between the two moon-faced ones, and both would be united in wedlock. Since what was proposed was not in accordance with what was fated, it chanced that the girl, out of curiosity that is ingrained in the behaviour of children, went on looking at that bowl and enjoyed the sight of water coming in through the hole. Suddenly, a single pearl, the size of a water drop, got detached from the bride's vestment and fell in the bowl and, rolling about, settled on the hole and stopped the inflow of water [into the bowl]. The astronomer waited for the prescribed moment, and the father sitting at another place similarly kept waiting. When the bowl did not function within the expected time, and the time passed, the father was astonished [exclaiming], 'O God, what has happened in secret that the bowl did not sink in the water!' When they really looked for the cause, they found that the single pearl had served as a stone-stop for water, and the moment they had been looking for had passed.

> [Four couplets]
> The father bit his fingers in disappointment…
> The loved one at that moment began to laugh
> The pearl from the window of the eye's bowl[19]

For one cannot afford to be angry with the star of Fate.

> What does the astrologer know of what exists behind the curtain?
> And as to who has drawn the painting upon this curtain?
> The geometrician, who has spent his life in pursuit of this art,
> Is in this painting a [mere] *muhra* (pawn/circle) drawn by a pair of compasses.

Finally, the luckless father told the ill-starred daughter: 'I shall write a book titled after your name, which will long endure in the world, for a good name is like a second life for one and confers immortal life upon the seed.'

Indeed, the book is a wonderful volume of writing, a unique narration. If the Greek observers of the movements of stars were to use it as a protective band on their arms, it would be just; and if the Persian experts of astronomical tables were to tie it as a talisman upon their heads, it would be appropriate.

[19] Meaning: a tear fell from the eye of the father.

It is like a bunch of flowers from the garden of science and knowledge, a work of art from the picture gallery of the precious and unique aspects of reality.

The translation of this work was undertaken by taking the help of the knowledge of the experts of this science, especially the astronomers of the Deccan. In the case of some Hindī (i.e., Sanskrit) terms whose equivalents were not found in other books dealing with this science, these were retained in their Hindī garb, and so explained that the language be not found difficult for a reader of Persian.

[Two couplets]
It is hoped that this book attains esteem
Becomes acceptable to the world as an aid to wish-fulfilment;
That from the aid of approval of the wisdom-promoting King,
It gets such a name that it becomes famous.

This book is so arranged as to consist of a Preface, a number of Rules and a Conclusion.

Appendix 2

*V*ipraputrīkathā[20]

मारुतवृन्दं जलदपटलीम् इव पुंसां विलसितं विधिर्विघटयति। यतः—

mārutavṛndaṃ jaladapaṭalīm iva puṃsāṃ vilasitaṃ vidhirvighaṭayati | yataḥ—

शक्यते नान्यथा कर्तुं धीमद्भिर्वैधसी लिपिः ।
विप्रपुत्र्याः शुभं लग्नं घटीरोधेऽन्यथाभवत् ॥

śakyate nānyathā kartuṃ dhīmadbhirvaidhasī lipiḥ |
vipraputryāḥ śubhaṃ lagnaṃ ghaṭīrodhe'nyathābhavat ||

तथा हि — वालासरग्रामवासिनः शब्दानुशासन-साहित्य-छन्दः-प्रभृति-प्रभूत-विद्या-नवद्यस्य विद्यानन्दस्य विप्रस्य विशेषतो ज्योतिःशास्त्रे वैदग्ध्यम् आसीत्। तस्य च वार्धके यमुनाकलत्रकुक्षिजा सुता जाता। अथ प्रवयाः स विप्रः स्वयं तस्या सुताया जन्मपत्रिकां विदधानः पतिसंज्ञस्य सप्तमस्थानस्य निर्णयं चिन्तयन् पाणिग्रहणानन्तरं षष्ठे मासे पतिमरणम् अज्ञासीत्। अहो मम सुतायाः शैशवे वैधव्यम् अस्ति। किं च नूनं जन्मपत्रीस्थिता हि ग्रहाः प्राक्तनं दैवं सूचयन्ति। दैवं हि सर्वत्र बलवत्तरम्। यतः —

[20] This is from the *Kathāratnākara* [KtRa1911, pp. 538–540].

tathā hi — vālāsaragrāmavāsinaḥ śabdānuśāsana-sāhitya-chandaḥ-prabhṛti-prabhūtavidyānavadyasya vidyānandasya viprasya viśeṣato jyotiḥśāstre vaidagdhyam āsīt | tasya ca vārdhake yamunākalatrakukṣijā sutā jātā | atha pravayāḥ sa vipraḥ svayaṃ tasyāḥ sutāyā janmapatrikāṃ vidadhānaḥ pati-saṃjñasya saptamasthānasya nirṇayaṃ cintayan pāṇigrahaṇānantaraṃ ṣaṣṭhe māse patimaraṇam ajñāsīt | aho mama sutāyāḥ śaiśave vaidhavyam asti | kiṃ ca nūnaṃ janmapatrīsthitā hi grahāḥ prāktanaṃ daivaṃ sūcayanti | daivaṃ hi sarvatra balavattaram | yataḥ —

यद्भग्नं धनुरीश्वरस्य शिशुना यज्जामदग्न्यो जितः
त्यक्ता येन गुरोर्गिरा वसुमती बद्धो यदम्भोनिधिः ।
एकैकं दशकन्धरस्य भयकृद्रामस्य किं वर्ण्यते
दैवं वर्णय येन सोऽपि सहसा नीतः कथाशेषताम् ॥

yadbhagnaṃ dhanurīśvarasya śiśunā yajjāmadagnyo jitaḥ
tyaktā yena gurorgirā vasumatī baddho yadambhonidhiḥ |
ekaikaṃ daśakandharasya bhayakṛdrāmasya kiṃ varṇyate
daivaṃ varṇaya yena so'pi sahasā nītaḥ kathāśeṣatām ||

`तस्य दैवस्य पुरस्तात् कैषा मामकीना सुता, कोऽहं च नृणां तृणम्' इति मत्वा सुखेन छात्राणां शास्त्राणि पाठयतस्तस्य सुता विवाहोचिता जाता। अथैकदा ज्योतिःशास्त्रं पाठयन् विद्यानन्दोऽस्मिन् लग्ने विवाहिता कन्यावश्यं वैधव्यं नाप्नोतीति लग्नं निर्णीय, `अहो मत्सुतायाः षष्ठे मासेऽवश्यं वैधव्यम्, इदं लग्नं चेदृशं यदत्र परिणीता कन्यावश्यं वैधव्यं नाप्नोतीति, तेन विलोकयामि, द्वयोर्मध्ये किं बलवत्तरम्' इति निर्णीय पित्रा कस्मैचिद् इभ्याय द्विजन्मसूनवे सुता वितीर्णा। अथाष्टादशदोषरहिते घटिकासाध्ये तस्मिन्नेव लग्ने पुत्री विवाहयितुम् आरब्धा (!)

'tasya daivasya purastāt kaiṣā māmakīnā sutā, ko'haṃ ca nṛṇāṃ tṛṇam' iti matvā sukhena chātrāṇāṃ śāstrāṇi pāṭhayatas tasya sutā vivāhocitā jātā | athaikadā jyotiḥśāstraṃ pāṭhayan vidyānando'smin lagne vivāhitā kanyāvaśyaṃ vaidhavyaṃ nāpnotīti lagnaṃ nirṇīya, 'aho matsutāyāḥ ṣaṣṭhe māse'vaśyaṃ vaidhavyam, idaṃ lagnaṃ cedṛśaṃ yadatra pariṇītā kanyā-vaśyaṃ vaidhavyaṃ nāpnotīti, tena vilokayāmi, dvayormadhye kiṃ balavat-taram' iti nirṇīya pitrā kasmaicid ibhyāya dvijanmasūnave sutā vitīrṇā | athāṣṭādaśadoṣarahite ghaṭikāsādhye tasminn eva lagne putrī vivāhayi-tumārabdhā (!) |

विशेषतो निःशेषज्योतिःशास्त्रकुशलो विनिर्मितकुङ्कुमतण्डुलतिलकः स विप्रः —

viśeṣato niḥśeṣajyotiḥśāstrakuśalo vinirmitakuṅkumataṇḍulatilakaḥ sa vipraḥ—

दशताम्रपलावर्तपात्रे वृत्तीकृते सति ।
घटिकायां समुत्सेधो विधातव्यः षडङ्गुलः ॥

विष्कम्भं तत्र कुर्वीत प्रमाणाद् द्वादशाङ्गुलम् ।
षष्ट्याम्भःपलपूरेण घटिका सद्भिरिष्यते ॥

daśatāmrapalāvartapātre vṛttīkṛte sati |
ghaṭikāyāṃ samutsedho vidhātavyaḥ ṣaḍaṅgulaḥ ||
viṣkambhaṃ tatra kurvīta pramāṇād dvādaśāṅgulam |
ṣaṣṭyāmbhaḥpalapūreṇa ghaṭikā sadbhiriṣyate ||

इत्यादिपरिपूर्णप्रमाणोपेतं घटिकापात्रं स्वच्छनीरभृते कुण्डे भगवतो भानोरस्तगमनसमये मुमोच। `मा कान्ते पक्षस्यान्ते पर्वाकशे <पर्याकाशे> देशे स्वाप्सीः' इत्यादि पलवृत्तपठनतो मनसो दुःसावधानतया पुत्रीपाणिग्रहणोत्सवौत्सुक्येन वार्धकेन च घटिकापात्रं पयःकुण्डे मुञ्चतः तस्यालिकतिलकात् पतन्न् एकः तण्डुलो घटिकारन्ध्रं रुरोध। रुद्धे च घटिकारन्ध्रे सा लाग्निकी वेला व्यतिचक्राम।

ityādiparipūrṇapramāṇopetaṃ ghaṭikāpātraṃ svacchanīrabhṛte kuṇḍe bhagavato bhānorastagamanasamaye mumoca | 'mā kānte pakṣasyānte parvākaśe <paryākāśe> deśe svāpsīḥ' ityādi-palavṛtta-paṭhanato manaso duḥsāvadhānatayā putrī-pāṇigrahaṇotsavautsukyena vārdhakena ca ghaṭikāpātraṃ payaḥkuṇḍe muñcataḥ tasyālikatilakāt patann ekaḥ taṇḍulo ghaṭikārandhraṃ rurodha | ruddhe ca ghaṭikārandhre sā lāgnikī velā vyaticakrāma |

तेन विप्रेणापि घटिकाभरणविलम्बेन तण्डुलजनितं घटिकारोधं ज्ञात्वेतिचिन्तितं, `अहो ज्योतिःशास्त्रं सत्यं, यतोऽमुष्मिन् लग्नेऽस्याः पाणिग्रहो नाभवत् वैधव्यस्यावश्यम्भावित्वात्', इति ध्यात्वा तेन गतेऽपि लग्ने पुत्री परिणायिता। षण्मासान्तरे च दन्दशूकदंशात् मृते भर्तरि सा विधवा जाता॥

tena vipreṇāpi ghaṭikābharaṇavilambena taṇḍulajanitaṃ ghaṭikārodhaṃ jñātveticintitam, 'aho jyotiḥśāstraṃ satyaṃ, yato'muṣmin lagne'syāḥ pāṇigraho nābhavat vaidhavyasyāvaśyambhāvitvāt,' iti dhyātvā tena gate'pi lagne putrī pariṇāyitā | ṣaṇmāsāntare ca dandaśūkadaṃśāt mṛte bhartari sā vidhavā jātā ||

इति विधिना लिखितं अन्यथा न स्याद् इत्यर्थे विप्रपुत्रीकथा ॥
iti vidhinā likhitam anyathā na syād ityarthe vipraputrīkathā ||

The story of a brahmin's daughter

Just as a multitude of winds scatters the masses of clouds, even so Fate (*vidhi*) disposes of the plans of men. For

It is not possible for intelligent persons to alter the writing of the Fate (*vaidhasī lipi*). The auspicious moment set for the marriage of the brahmin's daughter became otherwise due to the blockage of the water clock.

It so happened that a brahmin named Vidyānanda, resident of the Vālāsarā village, whose mastery of grammar, literature, prosody and other subjects was impeccable, was especially proficient in astral science (*jyotiḥśāstra*). A daughter was born to him, in his old age, by his wife Yamunā. Now this brahmin of advanced age, while casting the daughter's horoscope himself, considered the status of the seventh house which is related to the [native's] husband and saw that it indicated the husband's death in the sixth month after the marriage. 'Alas, my daughter is fated to be a widow in her childhood. But certainly, the planetary positions in the horoscope indicate what had already been fated a long time ago (*prāktanaṃ daivaṃ*). Fate is indeed stronger in all cases everywhere. For

> Why praise every single deed of Rāma, who taught fear to the ten-headed [Rāvaṇa]? As a mere child he broke the [mighty] bow of Śiva, he vanquished [the invincible Paraśurāma] the son of Jamadagni, he gave up the earth on his father's command, he girded the ocean [with a bridge] and so on. Praise instead Fate, which forcibly brought even him to a state where nothing remained of him except his story.

'Against such Fate what can my [little] daughter do? What can I do, I a mere blade of grass among men?' Having pondered thus, he continued, with ease of heart, to teach *śāstras* to his pupils until his daughter reached the age of marriage. Once, while teaching astral science, Vidyānanda found such an auspicious moment *(lagna)* that a girl married then would never become a widow. 'It is amazing! On the one hand, my daughter is fated to be a widow in the sixth month; on the other, the auspicious moment is such that a girl married at this moment will definitely not become a widow; now let me see which of the two is stronger.' Having thought thus, he offered the daughter in marriage to the son of a wealthy brahmin. Then he proceeded to marry her off at that auspicious moment, which is devoid of any of the 18 kinds of faults, and which has to be determined by the water clock (*ghaṭikā-sādhya*).

The brahmin, who is especially well-versed in the whole range of astral science, wore a forehead-mark made of saffron and rice-grains (i.e., a mark made of saffron paste on which two or three rice grains were stuck), took the bowl of the water clock (*ghaṭikā-pātra*) —

> The round vessel is made of ten *palas* of copper.
> In the bowl the height should be set at six *aṅgulas*.
> The diameter there should be made of the measure of twelve *aṅgulas*.
> The good cherish a water clock that holds sixty *palas* of water.

The brahmin took the bowl, which was made fully according to the aforementioned prescriptions, and placed it in a basin filled with clean water, at the time of the setting of the divine Sun. Because he was busy reciting the *pala*

verses (*pala-vṛttas*) such as *'mā kānte pakṣasyānte paryākāśe deśe svāpsīḥ'*,[21] because his mind was preoccupied, because he was excited about the festivities of his daughter's marriage, and because he was old, [he did not notice] that, when he was placing the bowl in the water basin, a grain of rice got loose from his forehead-mark, fell into the bowl, and blocked its hole. Since the hole in the water clock was blocked, the [actual] time of the auspicious *lagna* elapsed.

When the brahmin realised the delay in the filling of the bowl, he noticed the blockage of the water clock caused by the rice grain, and thought: 'Indeed, the astral science is inviolable, for her marriage did not take place in this auspicious moment because widowhood was sure to occur without fail.' Thinking thus, even though the auspicious moment had lapsed, he married the daughter off. After six months, the husband died from a snake bite and she became a widow.

Thus the story of the brahmin's daughter [is narrated here] to illustrate the maxim: Fate's writing cannot be altered (*vidhinā likhitam anyathā na syāt*).

[21] *Pala-ślokas* are verses consisting of 60 long syllables. One recitation of such a verse at an even speed should take 1 *pala* (= 24 seconds); therefore they are called *pala-vṛttas* or *pala-ślokas*. While the water clock measures a full *ghaṭī* of 60 *palas* (= 24 minutes), the *pala-ślokas* are recited to measure fractions of a *ghaṭī* [Sar2001] and [Sar2004b].

The poetical face of the mathematical and astronomical works of Bhāskarācārya

Pierre-Sylvain Filliozat*

Bhāskarācārya is basically a *paṇḍita* of the twelfth century, well-versed in Sanskrit *śāstras* at the level they had reached in his times. That was a period of great refinement, a summit in all cultural fields, science, literature and arts. The works left by him for posterity show him as a great *jyotiṣī* and a talented *kavi*. Mathematical and astronomical knowledge are used with great poetic skills together in his *Siddhāntaśiromaṇi*. That invites us to join literary analysis to mathematical study for proper elucidation of his work, and, considering his leaning for poetry, to resort to *alaṃkāraśāstra, chandas* and eventually to *vyākaraṇa*. *Alaṇkāraśāstra* teaches three levels of poetry, the ornamentation of words through alliterations etc., the mechanism of meaning ornamentation through comparison (*upamā*), metaphor (*rūpaka*), etc., and finally a superior level at which the sound of words and their meaning suggest ideas and sentiments in the mind of a sympathising audience (*sahṛdaya*). This top level is called *dhvani* i.e., prolongation in suggested meaning comparable to sound expanded in its *anuraṇana* "resonance". The highest *dhvani* is described as the instant communion of the audience in a suggested sentiment. It creates the state of *rasa* or "aesthetic savour" colored by diverse emotions and sentiments. The intent of this paper is to examine Bhāskara's application of such concepts in the composition of a scientific *jyotiṣa* text. For this purpose we propose very literal translations for all words and expressions of Bhāskara commented herein.

We start with the *maṅgalaśloka* of the *Līlāvatī* [Pat2004, p. 4]:

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प्रीतिं भक्तजनस्य यो जनयते विघ्नं विनिघ्नन् स्मृतः
तं वृन्दारकवृन्दवन्दितपदं नत्वा मतङ्गाननम् ।
पाटीं सद्गणितस्य वच्मि चतुरप्रीतिप्रदां प्रस्फुटां
संक्षिप्ताक्षरकोमलामलपदैर्लालित्यलीलावतीम् ॥

prītiṃ bhaktajanasya yo janayate vighnaṃ vinighnan smṛtaḥ
taṃ vṛndārakavṛndavanditapadaṃ natvā mataṅgānanam |
pāṭīṃ sadgaṇitasya vacmi caturaprītipradāṃ prasphuṭāṃ
saṃkṣiptākṣarakomalāmalapadairlālityalīlāvatīm ||

He who creates bliss for the devoted folk, dispelling obstacles, when thought of,
Him, whose feet are honoured by the troops of gods, the Elephant-faced, I salute.
Then I tell a procedure of good calculation, source of joy to the skilled, very clear,
by means of short, melodious, flawless words, playfully courteous.

This initial stanza contains first the traditional auspicious invocation of a god, then states the intent of the author, by giving the characteristics of the planned composition.

The tradition is to pronounce an auspicious stanza at the beginning, and sometimes at the middle and end of a work. Bhāskara has followed the custom with the invocation of a god at the beginning of the main divisions of his work. They are *Līlāvatī, Bījagaṇita, Grahagaṇitādhyāya* and *Golādhyāya.* Each subdivision has also an initial declaration of intent (*pratijñā*). Bhāskara thought it relevant to join an invocation in the case of the *bhuvanakośa* subsection of the *Golādhyāya.* He gave the most graceful one at the beginning of the *Līlāvatī* section after his exposition of the types of measures, which he conceived of as explanatory (*pāribhāṣika*) for the whole *Siddhāntaśiromaṇi.*

In the declaration of intent, each word characterises the *pāṭī*, which is the proper subject of the *Līlāvatī* section. *Sadgaṇitasya pāṭī* refers to "a place of display in totality of all calculations established rationally in common usage and in the *śāstra*", according to the words of Śaṅkara Vāriyar in his *Kriyākramakarī* commentary on the *Līlāvatī* [Līlā1975, p. 3]: *"***sato** *yuktisiddhasya laukikasya śāstrīyasyāpi nikhilasya gaṇitasya* **pāṭīṃ** *sākalyena saṃdarśanasthānabhūtām".*

Then three qualifications of the *pāṭī* follow. *Caturaprītipradām* "giver of felicity to the skilled" may refer to the pleasure of the reader of the poetical side of the work and to his pleasure in seeing the ingeniousness of the mathematical procedures. *Prasphuṭām* "very clear" refers to the facility of a correct understanding. *Lālityalīlāvatīm* is a qualifier of the *pāṭī.* Diverse English translations have been proposed for this expression. I deem all to be acceptable. Still, I think that one more shade of meaning can be extracted as *dhvani.* The word *līlā* "play" is used to refer to the great deeds of a god, such as the actions of creation etc. of the Supreme Lord. The concept is that for God the most difficult actions are extremely easy to perform. In all schools of *Vedānta*

the phenomenal world is called the *līlā* of *Īśvara* or *Parabrahman*. Vācaspati Miśra expresses it beautifully by telling that creating the *prapañca* is no less effort for the Supreme than smiling: *smitam etasya carācaram* "mobile and immobile creation for Him is a smile" [BrSū1997, v. 2, p. 3]. The *pāṭī* text composed by Bhāskara is *līlāvatī* in the sense that it is "playful", i. e. making mathematics easy like a play. The *Līlāvatī* is the smile of Bhāskara.

This playfulness itself is qualified by *lālitya* "courteouness, elegance". In Sanskrit dramaturgy (*nāṭyaśāstra*) a type of hero is called *dhīra-lalita* "constantly courteous". It is the gallant hero who courts the heroine. Now for every rule Bhāskara gives examples (*udāharaṇa*) and he has composed their formulation in a form fit for enacting according to the conventions of Sanskrit theatre.[1] Each example is a question, a request for the solution of a problem. The questioner is the mathematician who teaches the rules. He addresses diverse characters, a child or a young girl (stanzas 14, 18, 51, 73), a loving girl (60, 77, 124), a male friend (*passim*), another mathematician (90, 145, 161, 166, 174, 184, 238, 261), a merchant (81, 92, 107, 111) and so on. The choice of the addressee is generally done according to the nature or difficulty of the problem. In all cases it is done amiably, gallantly. The mathematician is a *dhīra-lalita* character, patting a child, courting or teasing a young playful girl, conversing with an intimate friend, addressing somebody in common life. Every problem is an example of *lālitya*.

How did Bhāskara obtain those three qualities? He says *saṃkṣiptākṣarakomalāmalapadaiḥ* "by means of words of restrained [numbers of] letters, melodious and flawless". That can be construed with all the qualifications of the *pāṭī*, on both sides, according to the principle of the central jewel in a necklace (*madhyamaṇi*). These qualifications apply to the rules as well as to the examples. The rules obey the principles of the *sūtra* or *kārikā* style and the first principle is brevity of expression. The style of the examples is more free. It is the place of all poetical expansions inspired by the *lālitya* of the mathematician. This is referred to by the qualification *komala* "sweet" which concerns the sound and the meaning of words. The term *amala* "flawless" speaks by itself. Precision and purity of language is another principle of *lālitya*.

The same qualification *lālityalīlāvatī* comes again for the poet's speech in the *maṅgalaśloka* of the *Golādhyāya* [SiSi1988, v. 1]:

सिद्धिं साध्यमुपैति यत्स्मरणतः क्षिप्रं प्रसादात्तथा
यस्याश्चित्रपदा स्वलङ्कृतिरलं लालित्यलीलावती ।

[1] Several examples of mathematical problems imagined by Bhāskara have been effectively brought to the stage by the talented dance master, Shri Venugopal Rao Sakaray, and enacted by students of the Rishi Valley School, on the 20th of September 2014, in an evening session of the Bhāskara 900 Conference held in the Vidya Prasarak Mandal at Thane, Mumbai.

नृत्यन्ती मुखरङ्गगेव कृतिनां स्याद्भारती भारती
तं तां च प्रणिपत्य गोलममलं बालावबोधं ब्रुवे ॥

siddhiṃ sādhyamupaiti yatsmaraṇataḥ kṣipraṃ prasādāttathā
yasyāścitrapadā svalaṅkṛtiralaṃ lālityalīlāvatī |
nṛtyantī mukharaṅgageva kṛtināṃ syādbhāratī bhāratī
taṃ tāṃ ca praṇipatya golamamalaṃ bālāvabodhaṃ bruve ||

A project comes fast to realisation by just thinking of Him; and by the grace of Her, with beautiful dance steps-like words, very ornate, courteously playful, like a dancing actress, going to the stage-like mouth of the learned, will be their speech, to Him and to Her I bow; then I tell the Sphere, flawlessly, accessibly to novices.

Again the speech of the *paṇḍita* is described in words of the world of theatre. The mathematician here again is a *dhīra-lalita* hero. His speech is characterized by the same courteousness. Here refinement of language goes a step further. The idea is expressed through a comparison (*upamā*) confirmed by an amalgam of two meanings in a single word (*śleṣa*) and a metaphor (*rūpaka*). The compared object is the speech, the comparing one is the dancer. The idea of comparison is expressed by the particle *iva,* which has been moved from the word *bhāratī* "actress" which it really qualifies, to the word *mukharaṅgagā*. The properties common to both terms of the comparison are signified by five qualifications. In *citra-padā* the single word *pada* means 'word' and 'dance step'. In *sv-alaṅkṛtiḥ, alaṅkṛti* refers to the literary 'ornaments' of the poet's speech and to the 'jewels' worn by the dancer. *Lālityalīlāvatī* refers to courteous language and graceful movements. *Nṛtyantī* refers to speech in verses, i.e., codified by *chandas* and, rhythmically recited, then to choreographed dance. In *mukha-raṅga-gā, mukha-raṅga* "mouth-stage" is a *rūpakasamāsa* 'metaphoric compound'. In *alaṅkāraśāstra* a metaphor is considered as a common property since it amalgamates two objects: the mouth of the poet is a stage for dance. The reason for the displacement of the particle *iva* is the *chandas*. The same word *bhāratī, bharatasya iyam* "relevant to the actor" is repeated to refer first to the poet's speech, then to the actress. It follows also that there is a type of alliteration called *lāṭānuprāsa* "alliteration of *Lāṭas*" which consists in the repetition of a word with a difference of intent.

One more comment on this verse should be made. It joins an invocation of Sarasvatī to an invocation of Gaṇeśa. A ritual, current in the Cālukyan period, enjoins the priest to invoke Gaṇeśa on the southern side, Sarasvatī on the northern side, Lakṣmī in the centre of the lintel of the door of the temple, at the time of entering to perform morning worship [Bru1963, I, p. 93]. There are temples of this period in west and south India bearing a sculptured representation of these deities, for instance on a stray lintel on Hemakūṭa at Hampi. The temple is conceived of as the universe of which the worshipped god is the Supreme Lord. At the time of describing the entire sphere (*gola*), entering the temple of the universe Bhāskara executes this double invocation.

This stanza strikes the mind of the reader by the elevated character of the concept. Bhāskara has couched it in a typical structure of Sanskrit poetry, as codified in *alaṅkāraśāstra*. A few other stanzas too follow this model, like the final one of the *Līlāvatī*, which is one more elegant description of the *pāṭī* amalgamated (*śliṣṭa*) with the description of a woman. This is enough to rank Bhāskara among creators in the world of Sanskrit *kāvya*. There is more in the *Siddhāntaśiromaṇi*. So far we have shown poetry in stanzas external to the contents of the work. There is also a blend of poetry in the mathematical and astronomical expositions.

In Bhāskara's compositions, two different styles are noticeable: *sūtra* and *udāharaṇa*. The *sūtra* is the formalised exposition of a scientific concept, a procedure of operation, a statement of a natural fact etc. Bhāskara has not gone to the same level of compactness and symbolism as Pāṇini or Piṅgala. But he has taken care to be short and precise in expression. He has his devices of shortening. The use of *bhūtasaṃkhyā*, metonymic representation of a number by an object characterised by it, allows him to insert in a verse a shorter expression. For instance, in the *Bījagaṇita*, the half-verse presenting the operations with zero [Pat2004, v. 14, p. 214]:

वधादौ वियत् खस्य खं खेन घाते
खहारो भवेत् खेन भक्तश्च राशिः ॥ १४ ॥

vadhādau viyat khasya khaṃ khena ghāte
khahāro bhavet khena bhaktaśca rāśiḥ || 14 ||

In multiplication etc. of zero: zero; in multiplication by zero: zero; a number divided by zero will be 'having a divisor zero'.

Two devices of shortening are used here: ellipsis, since the word *rāśi* 'number' has been mentioned only once and is understood in two expressions; metonymic expression of zero by the words *viyat* or *kha* meaning 'void space' or 'sky'. This is not natural language. It is a type of formalization which is purely oral and remains concrete. There is no use of graphic devices, nor the completely abstract form of modern formalization:

$$0 \times n = 0;\ \frac{0}{n} = 0; \quad n \times 0 = 0;\ \frac{n}{0} = n/khahāra.$$

To the question: where is poetry here? One may answer that there is alliteration of the sound *kh* and gutturals in the mould of the *bhujañgaprayāta* metre "serpent's walk". There is also the concrete mode of expression. *Kha*, literally "void space", mentioned to refer to a number, is an instance of the *lakṣaṇāvṛtti* "indirect mode of expression" which is the foundation of many poetical ornaments. Bhāskara has gone further in a comparison of the *khahara* with the Supreme Lord. Following Sūryadāsa in his commentory *Sūryaprakāśa* on

the *Bījagaṇita* [Pat2004, v. 16, p. 127], we can say that "Now, he shows that in the science of calculation there is another name for the *khahara* number, namely 'infinite'. And he describes its being infinite with a proof".

> अथ गणितशास्त्रे खहरस्याङ्कस्य संज्ञान्तरम् अस्तीति प्रकटयत्यनन्त इति । अथ तस्यानन्तत्वं युक्त्या निरुपयति —
>
> *atha gaṇitaśāstre khaharasyāṅkasya saṃjñāntaram astīti prakaṭayatyananta iti | atha tasyānantatvaṃ yuktyā nirūpayati —*

> अस्मिन्विकारः खहरे न राशावपि प्रविष्टेष्वपि निःसृतेषु ।
> बहुष्वपि स्याल्लयसृष्टिकालेऽनन्तेऽच्युते भूतगणेषु यद्वत् ॥ २० ॥
>
> *asminvikāraḥ khahare na rāśāvapi praviṣṭeṣvapi niḥsṛteṣu |*
> *bahuṣvapi syāllayasṛṣṭikāle'nante'cyute bhūtagaṇeṣu yadvat* || 20 ||
>
> In this *khahara*, though being a number, no change, even when many enter and go out, should occur, as it is at the time of creation and destruction when troops of beings [enter in and go out of] Infinite Acyuta.

We understand here that the proof of the infinite nature of the *khahara* number is the fact that it does not change when something is added or subtracted, which is the property of infinity. This is a mathematical fact. The *khahara* is a fraction with denominator zero. When adding or subtracting another number, after reduction to the same denominator it does not change:

$$\frac{a}{0} \pm b = \frac{a \pm (b \times 0)}{0} = \frac{a}{0}.$$

Bhāskara derives the idea of the *khahara* being infinity, from this absence of change. Then in an elevated flight of imagination he compares the *khahara* to the Supreme Lord, on the basis of their common property of immutability. He derives the infinity of the *khahara* from the mathematical fact, not from the comparison. Sūryadāsa, after giving this explanation in detail, distinguishes correctly the mathematical *yukti* and the poetical expansion, when he says: "then strengthening with the example of Viṣṇu, because of the common property of the *khahara* to be infinite, he shows the high savour of his poetry".

> अथ खहरस्यानन्तत्वसाधर्म्यात् विष्णुदृष्टान्तेन दृढयन् स्वकविताचमत्कारं दर्शयति यद्वदिति ।
>
> *atha khaharasyānantatvasādharmyāt viṣṇudṛṣṭāntena dṛḍhayan*
> *svakavitācamatkāraṃ darśayati yadvaditi |*

Once Bhāskara has derived the infinite nature of *khahara* from its immutability, he lets his imagination expand freely.[2] In this example we appreciate the relevance of the comparison to the mathematical reality. The most

[2] We express our thanks to Profs. K. Ramasubramanian, M. D. Srinivas and scholars who cleared our doubts by their remarks and giving us relevant references to the *Vāsanā*

fitting poetry closely approaches the scientific thought, while the difference of register is preserved. We presume also that the personality of Bhāskara is such that he has an equal inclination to rational thinking and poetical inspiration. As a mathematician he enjoys the beauty of the logic and of discovering hidden facts. As a poet he enjoys the beauty of the appropriate image.

The *udāharaṇa* style is differentiated from the *sūtra* style by the absence or moderate degree of formalisation and by its closeness to concrete situations. That is profusely exemplified in the *Līlāvatī*. There is conciseness only under the constraint of the metre and at the same time no prolixity. On a moderate pace, in the mould of diverse metres, *upajāti, vasantatilaka, śārdūlavikrīḍita, āryā, gīti* etc., without rare vocabulary, Bhāskara expresses facts precisely and elegantly. The care for *lālitya* is always present. He uses diverse conventions of Sanskrit *kāvya* and themes, devices of *alaṅkāraśāstra* always with moderation. The result is a charm felt by any reader, whatever is his interest and purpose, scientific or literary.

The apex of poetry is *rasa*, i. e. the aesthetic pleasure experienced by the reader when sharing a suggested mood or sentiment. Bhāskara creates a *rasa* by placing the elements of a problem of mathematics in a concrete situation apt at suggesting a particular mood. In a problem of *iṣṭakarma* a broken necklace suggests the *saṃbhoga* "love in union" side of *śṛṅgāra-rasa* "amorous mood":[3]

---

etc. In his auto-commentary Bhāskara gives himself another derivation of his concept of *khahara* as infinity [BīGa2008].

> खहरश्च राशिरनन्तसमः । कस्मिंश्चित् स्थिरभाज्ये उत्तरोत्तरम् अल्पहारेण भक्ते लब्धिरुत्तरोत्तर-मधिका । एवमत्र परमाल्पेन शून्यसमेन हारेण विभाजिते लब्धिः अनन्तसमा ।
>
> *khaharaśca rāśiranantasamaḥ | kasmiṃścit sthirabhājye uttarottaram alpahāreṇa bhakte labdhiruttarottaram adhikā | evam atra paramālpena śūnyasamena hāreṇa vibhājite labdhiḥ anantasamā |*
>
> The number having a divisor zero is equal to infinity. When a fixed dividend is divided by a progressively smaller divisor the quotient is progressively greater. Thus when it is divided by the extremely small divisor equal to zero the result is equal to infinity.

However, the fact that this passage is not found in all editions of the *Vāsanā* auto-commentary of Bhāskara, is to be considered.

[3] This problem is already found in the *Triśatikā* of Śrīdhara. It is given with a different formulation in the *Gaṇitasārasaṃgraha* of Mahāvīra. The stanza, as composed by Śrīdhara, has crept into a few editions of the *Līlāvatī*, with a variant reading [Sar2004a, pp. 463–476]. It is probably an interpolation, maybe of early date, since a commentator has inserted it in his commentary.

कामिन्या हारवल्ल्याः सुरतकलहतो मौक्तिकानां त्रुटित्वा[4]
भूमौ यातस्त्रिभागः शयनतलगतः पञ्चमांशोऽस्य दृष्टः ।
प्राप्तः षष्ठः सुकेश्या गणक दशमकः संगृहीतः प्रियेण
दृष्टं षट्कं च सूत्रे कथय कतिपयैर्मौक्तिकैरेष हारः ॥ ५३ ॥

kāminyā hāravallyāḥ suratakalahato mauktikānāṃ truṭitvā
bhūmau yātastribhāgaḥ śayanatalagataḥ pañcamāṃśo'sya dṛṣṭaḥ |
prāptaḥ ṣaṣṭhaḥ sukeśyā gaṇaka daśamakaḥ saṃgṛhītaḥ priyeṇa
dṛṣṭaṃ ṣaṭkaṃ ca sūtre kathaya katipayairmauktikaireṣa hāraḥ || 53 ||

The necklace of the young girl in the struggle of pleasure having broken, of the pearls one third went on the ground, one fifth was seen going on the bed, one sixth was taken by her of beautiful hair, O mathematician, one tenth was collected by her lover, one sixth was seen on the thread; tell how many pearls this necklace had.

The process of suggestion (*dhvani*) of *rasa* is achieved by the disorder of the pearls. At the same time an ingenious method leads to the solution. Supposing the number of pearls is 1, the scattered pearls are

$$1 - (\frac{1}{3} + \frac{1}{5} + \frac{1}{6} + \frac{1}{10}) = 1 - \frac{48}{60} = \frac{1}{5}.$$

The total number of pearls in the necklace is $\frac{6\times 1}{\frac{1}{5}} = 30$.

Bhāskara loved nature, especially the fauna, to which he ascribed in the experience of human sentiments. With the conventions of Sanskrit poets he has imagined natural scenes presenting mathematical problems and suggesting experiences of diverse *rasas* at the same time. In a section called *guṇakarma* he deals with a class of problems of finding a number of which the sum or difference with a multiple of its square root is known: $x^2 \pm mx = a$ in modern notation. Bhāskara has been inspired by this structure of a group in which one part is unknown, the other known. He appropriately imagines examples of same structure: the number of elements of a group is unknown; in one part the number of elements is also unknown; in another part that number is known. He describes groups of animals which are differentiated in subgroups by their diverse occupations. The first example is a story of swans in a pond [Līlā1937, vol. 1, v. 67, p. 64]:

बाले मरालकुलमूलदलानि सप्त
तीरे विलासभरमन्थरगान्यपश्यम् ।
कुर्वच्च केलिकलहं कलहंसयुग्मं
शेषं जले वद मरालकुलप्रमाणम् ॥ ६७ ॥

[4] See [Līlā1937, vol. 1, p. 17]. The editor has placed this stanza in the commentary of Gaṇeśa, not in the main text of Bhāskara. It has been recognized as pertaining to the text of the *Līlāvatī* in [Līlā2001, p. 58], with a cruder variant reading of the first quarter of the stanza: *hāras tāras taruṇyā nidhuvanakalahe mauktikānāṃ viśīrṇaḥ*.

bāle marālakulamūladalāni sapta
tīre vilāsabharamantharagāṇyapaśyam |
kurvacca kelikalahaṃ kalahaṃsayugmaṃ
śeṣaṃ jale vada marālakulapramāṇam || 67 ||

Child, seven times the half of the square root of a troop of swans, on the bank, slowed down by the fatigue of their enjoyments, I have seen; giving itself to an amorous quarrel a couple of pretty swans remains in the water; tell the size of the troop of swans.

The process of solving the problem is to take the square of half the multiplicator $\frac{7}{4}$, that is $\frac{49}{16}$, add to it the known quantity 2, that is $\frac{81}{16}$, whose square root $\frac{9}{4}$ is added to the half of the multiplicator $\frac{7}{4}$, that is $\frac{16}{4} = 4$. The square of this result 16 is the size of the troop. In this problem, the unknown group is the largest one, is of complex structure and is subject to a "play" of successive arithmetical actions. The known group is small and subject to one involvement. The poet appears to have carefully adapted his example to the problem, when he divided the flock of birds in a large group tired of too many "plays" and a happy couple. Here again, *śṛṅgāra-rasa* is suggested in two moments of the *saṃbhoga* aspect.

Then, Bhāskara adds one more element to the problem. The number to find is reduced to square root and a fraction. The whole group has therefore three parts, two unknowns and one known. Bhāskara illustrates it with three examples, deserving to be quoted here for their poetical charm. They are stories of animals and of a *Mahābhārata* hero. One more describes the diverse actions of swans or more exactly *haṃsas*. In Sanskrit poetry the *haṃsa* is a natural bird with a mythological dimension, a symbol of purity and discrimination [Līlā1937, vol. 1, v. 69] :

यातं हंसकुलस्य मूलदशकं मेघागमे मानसं
प्रोड्डीय स्थलपद्मिनीवनमगादष्टांशकोऽम्भस्तटात् ।
बाले बालमृणालशालिनि जले केलिक्रियालालसं
दृष्टं हंसयुगत्रयं च सकलां यूथस्य संख्यां वद ॥६९॥

yātaṃ haṃsakulasya mūladaśakaṃ meghāgame mānasaṃ
proḍḍīya sthalapadminīvanamagādaṣṭāṃśako'mbhastaṭāt |
bāle bālamṛṇālaśālini jale kelikriyālālasaṃ
dṛṣṭaṃ haṃsayugatrayaṃ ca sakalāṃ yūthasya saṃkhyāṃ vada || 69 ||

Ten times the square root of a flock of *haṃsas* went to Mānasa lake at the advent of clouds; one eighth of it flew away from the bank of water towards a garden of ground-lotuses; O little girl, in the water full of fresh lotus stalks, longing for amorous sports, one sees three couples of *haṃsas*; tell the full number of the flock.

The solution is 144 birds. The migration to the Mānasa lake is a symbol of the detachment of the *saṃnyāsin*. The plays in water represent the attachment to the world. *Śānti rasa* "appeasement mood" is suggested by the flight of the

first group, *śṛṅgāra-rasa* by the sports of the remaining group. This beautiful stanza reminds us also of the famous vedic image of two birds on the same tree, one eating the fruits, the other looking around without eating [Muṇ1999, 3.3.1]:

द्वा सुपर्णा सयुजा सखाया समानं वृक्षं परिषस्वजाते ।
तयोरन्यः पिप्पलं स्वाद्वत्त्यनश्नन्नन्यो अभिचाकशीति ॥

dvā suparṇā sayujā sakhāyā samānaṃ vṛkṣaṃ pariṣasvajāte |
tayor anyaḥ pippalaṃ svādvattyanaśnannanyo abhicākaśīti ||

Two birds ever united, of same stock, cling to the same tree. One of them eats the sweet fruit, the other, not eating, looks on.

The largest part of Sanskrit poetry and theatre is shared between *śṛṅgāra-rasa* and *vīra-rasa* the heroic mood. Bhāskara calculates the number of arrows discharged by Arjuna to kill Karṇa. With the distribution of one hundred he suggests the *vīra-rasa* [Līlā1937, vol.1, v. 70] :

पार्थः कर्णवधाय मार्गणगणं क्रुद्धो रणे सन्दधे
तस्यार्धेन निवार्य तच्छरगणं मूलैश्चतुर्भिर्हयान् ।
शल्यं षड्भिरथेषुभिस्त्रिभिरपिच्छत्रं ध्वजं कार्मुकं
चिच्छेदास्य शिरः शरेण कति ते यानर्जुनः सन्दधे ॥ ७० ॥

pārthaḥ karṇavadhāya mārgaṇagaṇaṃ kruddho raṇe sandadhe
tasyārdhena nivārya taccharagaṇaṃ mūlaiścaturbhirhayān |
śalyaṃ ṣaḍbhiratheṣubhistribhirapicchatraṃ dhvajaṃ kārmukam
cicchedāsya śiraḥ śareṇa kati te yānarjunaḥ sandadhe || 70 ||

Pārtha in order to kill Karṇa discharged a mass of arrows, furious as he was in battle; after diverting the mass of his opponent arrows with one half and Śalya with six arrows, he cut off the horses with four times the square root, then the umbrella, the standard, the bow with three and the head of Karṇa with one arrow; how many were the arrows that Arjuna discharged.

The first half of the stanza gives the unknown quantities $\frac{1}{2}x^2$ and $4x$; the second half gives the known quantities 6, 3, 1. Their total 10 is involved in the calculation. Gaṇeśa and Gaṅgādhara have taken the word *śalyam* as referring to Śalya, king of Madras and charioteer of Karṇa. They link him with *nivārya* to obtain the meaning: “diverting Śalya out of the battle”, since Śalya survives the battle between Arjuna and Karṇa, which is fatal for Karṇa. The word *atha*, ‘then’, expected at the beginning of the second half of the stanza, has been displaced for the sake of the *śārdūlavikrīḍita* metre.

Finally comes a story of bees [Līlā1937, vol.1, v. 71] :

अलिकुलदलमूलं मालतीं यातमष्टौ
निखिलनवमभागाश्चालिनी भृङ्गमेकम् ।

निशि परिमललुब्धं पद्ममध्ये निरुद्धं
प्रतिरणति रणन्तं ब्रूहि कान्तेऽलिसंख्याम् ॥ ७१ ॥

alikuladalamūlaṃ mālatīṃ yātamaṣṭau
nikhilanavamabhāgāś cālinī bhṛṅgamekam |
niśi parimalalubdhaṃ padmamadhye niruddhaṃ
pratiraṇati raṇantaṃ brūhi kānte'lisaṃkhyām || 71 ||

The square root of a half of a swarm of bees and $\frac{8}{9}^{ths}$ of the whole have gone to *mālatī* flowers; a female bee responds to one humming male who out of greed for fragrance has been caught inside a lotus [closing] at night. O beloved, tell the number of bees.

Two parts of the swarm are described. The first one evokes the major part of swarm in its movement over the flowers and that is expressed with a noteworthy alliteration of *'l'*. The second part consists in two bees, a couple, male and female. The alliteration in *'l'* is balanced by an alliteration in *'r'*, evoking the respective humming of the male and the female. The humming of the male is presented as due to the greed for the pollen. The humming of the female evokes her hovering over the lotus. That unfailingly calls to mind a theme of Sanskrit poetry, the separation of a couple, because the husband has gone for trade out of greed and the wife remains alone. That is the *vipralambha* "love in separation" aspect of *śṛṅgāra-rasa*. The bipartition of the scene appropriately corresponds to the two sides of the equation involved in the problem, in modern notation $x^2 - \sqrt{\frac{x^2}{2}} - \frac{8}{9}x^2 = 2$.

Pāṭī is concerned mostly with arithmetic, but deals also with mensurations of sides of geometrical figures. It was illustrated with abstract designs, as can be seen in manuscripts. Pursuing his method of illustrating problems with concrete situations, Bhāskara has imagined scenes organised in the space of geometrical figures. For instance [Līlā1937, vol. 2, v. 152]:

अस्ति स्तम्भतले बिलं तदुपरि क्रीडाशिखण्डी स्थितः
स्तम्भे हस्तनवोच्छ्रिते त्रिगुणितस्तम्भप्रमाणान्तरे ।
दृष्ट्वाहिं बिलमाव्रजन्तमपतत् तिर्यक् स तस्योपरि
क्षिप्रं ब्रूहि तयोर्बिलात् कतिकरैः साम्येन गत्योर्युतिः ॥ १५२ ॥

asti stambhatale bilaṃ tadupari krīḍāśikhaṇḍī sthitaḥ
stambhe hastanavocchrite triguṇitastambhapramāṇāntare |
dṛṣṭvāhiṃ bilamāvrajantamapatat tiryak sa tasyopari
kṣipraṃ brūhi tayorbilāt katikaraiḥ sāmyena gatyoryutiḥ || 152 ||

There is at the foot of a pillar a hole and at the top stands a pet peacock; the pillar being nine cubits high, at a distance three times the size of the pillar. Seeing a serpent proceeding towards the hole, the peacock pounced upon it obliquely. Quickly tell at how many cubits from the hole will they meet, by assuming the equality of their two movements?

The answer is 12 cubits. A manuscript said to be dated 1650 CE besides the text also gives sketches of geometrical figures. The present stanza appears at

the bottom of the recto of a folio, the solution on top of the verso. The scribe was probably more interested in the story than in the problem, because instead of drawing a relevant geometrical figure, as he has done for the previous and the next problem, he has done a nice drawing of a peacock on a pillar with a serpent at the bottom.

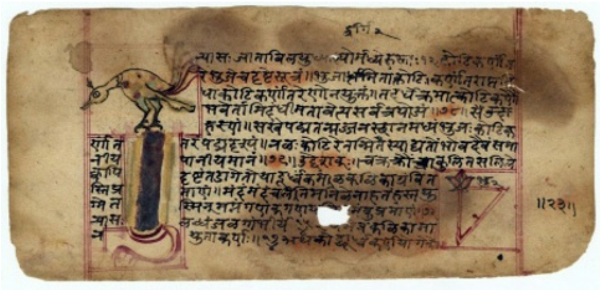

Figure 1: The peacock and serpent problem in a manuscript.

The same stanza has been excellently enacted by dance students of Rishi Valley School.

Figure 2: Enactment of the peacock and snake problem in Rishi valley school.

There is one more level of poetry in which Bhāskara excels, that is the description of the universe. Indian scientific astronomy has accepted elements

of the *purāṇic* cosmography in its representation of time and space. That is shown by the concepts of diverse time measures and spatial residences for gods, demons, *pitṛs* etc. as well as for men. Bhāskara has received the traditional teaching of his time and brought into it his own perceptions of scientist and poet. That is clear in its *kālamānādhyāya* "lesson on measurement of time" in the *Grahagaṇita* section. He recalls the concept of creation of earth, stars and planets, the setting in motion of the moving celestial bodies simultaneously at a fixed point of time, the beginning of a *Mahāyuga*. He recalls the long time divisions of Brahmā, Gods, *Pitṛs*, etc. As a scientist he connects them to the observable periods of men on the earth. He follows Brahmagupta in accepting the traditional decreasing durations of the four *yugas*, from *Kṛta* to *Kali*, but not Āryabhaṭa who makes them equal. He takes the beginning of the *Mahāyuga* as epoch for the simultaneous start of the revolutions of celestial bodies. It follows that during this period the planets do an integral number of revolutions, before taking a new start. Here Bhāskara declares that for the purpose of calculations of the position of planets, it is not necessary to speculate further in which larger period, *manvantara* or *kalpa*, the present day is situated. He remains more moderate on this point than other authors of *jyotiṣa* and tells them with courteous humour [SiŚi1939, vol. I, vv. 26–27]:

तथा वर्तमानस्य कस्यायुषोऽर्धं गतं सार्धवर्षाष्टकं केचिदूचुः ।
भवत्वागमः कोऽपि नास्योपयोगो ग्रहा वर्तमानद्युयातात्प्रसाध्याः ॥ २६ ॥
यतः सृष्टिरेषां दिनादौ दिनान्ते लयस्तेषु सत्स्वेव तच्चारचिन्ता ।
अतो युज्यते कुर्वते तां पुनर्येऽप्यसत्स्वेषु तेभ्यो महद्भ्यो नमोऽस्तु ॥२७॥

tathā vartamānasya kasyāyuṣo'rdhaṃ
gataṃ sārdhavarṣāṣṭakaṃ kecidūcuḥ |
bhavatvāgamaḥ ko'pi nāsyopayogo
grahā vartamānadyuyātātprasādhyāḥ || 26 ||
yataḥ sṛṣṭireṣāṃ dinādau dinānte
layasteṣu satsveva taccāracintā |
ato yujyate kurvate tāṃ punarye
'pyasatsveṣu tebhyo mahadbhyo namo'stu || 27 ||

Some have told that half of the present age of Brahmā has gone, or eight and a half of his years. Let there be some tradition. There is no utility of it. The positions of planets are to be calculated in the present passing age. Because there is creation of them at the beginning of an age and repose at the end, reflection on their movements, when they exist, is valid. To those great scholars who still do it when they do not exist, let there be a salute.

A Sanskrit poet draws a lot of his inspiration from the Indian myths. Bhāskara is a Sanskrit poet, but he has tempered his *pāṇḍitya* by its scientific purpose, tempering at the same time his scientific exposition by his *lālitya*.

The same can be said of the concept of space as described in the *Bhuvanakośa* division of the *Golādhyāya*. Bhāskara endorses the tradition, con-

nects it to scientific observation and describes it as a real poet. He criticises with slight humour those who did theories which are exaggerated or deviating from reasonable deductions. A quotation of a few stanzas on the position of the earth shows it clearly. At the same time Bhāskara offers a grandiose vision, with an outpour of his personal feeling of intense wonder [SiSi1988, vv. 2–5]:

> भूमेः पिण्डः शशाङ्कज्ञकविरविकुजेज्यार्किनक्षत्रकक्षा-
> वृत्तैर्वृत्तो वृतः सन् मृदनिलसलिलव्योमतेजोमयोऽयम् ।
> नान्याधारः स्वशक्त्यैव वियति नियतं तिष्ठतीहास्य पृष्ठे
> निष्ठं विश्वं च शश्वत् सदनुजमनुजादित्यदैत्यं समन्तात् ॥ २ ॥
>
> *bhūmeḥ piṇḍaḥ śaśāṅkajñakaviravikujejyārkinakṣatrakakṣā-*
> *vṛttairvṛtto vṛtaḥ san mṛdanilasalilavyomatejomayo'yam |*
> *nānyādhāraḥ svaśaktyaiva viyati niyataṃ tiṣṭhatīhāsya pṛṣṭhe*
> *niṣṭhaṃ viśvaṃ ca śaśvat sadanujamanujādityadaityaṃ samantāt* || 2 ||
>
> The mass of the earth, being round, encircled by the orbits of the Moon, Mercury, Venus, Sun, Mars, Jupiter, Saturn and the stars, being made of earth, wind, water, void, fire, having no support other [than itself], by its own force stands in space firmly; on its back eternally everything is placed including the sons of Danu, of Manu, of Aditi and Diti everywhere.
>
> सर्वतः पर्वतारामग्रामचैत्यचयैश्चितः ।
> कदम्बकुसुमग्रन्थिः केसरप्रसरैरिव ॥ ३ ॥
>
> *sarvataḥ parvatārāmagrāmacaityacayaiścitaḥ |*
> *kadambakusumagranthiḥ kesaraprasarairiva* || 3 ||
>
> On all sides it is covered with multitudes of hills, groves, villages, temples, like the tight knot of the flower of *kadamba* with its multitudes of filaments.

The *kadamba* flower, *Lamarckiana cadamba*, is here the most wonderfully appropriate standard of comparison. This image has been initially imagined by Āryabhaṭa and expressed in his forceful style in a *āryā* stanza [AB1976, ch. 4.7]. Bhāskara has rewritten the stanza in a shorter *anuṣṭubh* form with alliterations and with precise words, adding his elegance of formulation.

> मूर्तो धर्ता चेद्धरित्र्यास्ततोऽन्यस्तस्याप्यन्योऽस्यैवमत्रानवस्था ।
> अन्त्ये कल्प्या चेत् स्वशक्तिः किमाद्ये किं नो भूमेः साष्टमूर्तेश्च मूर्तिः ॥ ४ ॥
>
> *mūrto dhartā ceddharitryāstato'nyastasyāpyanyo'syaivamatrānavasthā |*
> *antye kalpyā cet svaśaktiḥ kimādye kiṃ no bhūmeḥ sāṣṭamūrteśca mūrtiḥ* || 4 ||
>
> If the earth had a solid support, then another [should support it], then another; thus a *regressus ad infinitum*. If one presumes its own force for the last support, why not for the first one? Why not for the earth? It is one body of the eight-bodied Śiva.
>
> यथोष्णतार्कानलयोश्च शीतता विधौ द्रुतिः के कठिनत्वमश्मनि ।
> मरुच्चलो भूरचला स्वभावतो यतो विचित्रा बत वस्तुशक्तयः ॥ ५ ॥

Figure 3: The filaments of the *kadamba* flower, *Lamarckiana cadamba*.

yathoṣṇatārkānalayośca śītatā
vidhau drutiḥ ke kaṭhinatvamaśmani |
maruccalo bhūracalā svabhāvato
yato vicitrā bata vastuśaktayaḥ || 5 ||

As there is heat in the sun and fire, cold in the moon, liquidness in water, hardness in stone, as wind is mobile, the earth is immobile by her own nature, because beautiful are the powers of things.

The tools of the Sanskrit poet are used by Bhāskara here: varied prosody, alliterations, comparisons in chain. We have to emphasize the inspired character of this passage. We feel the enthusiasm of the poet astronomer, dragged by the beauty of the universe, carried away by his discovery of its divinely built structure. A saying of western astronomers in Latin is: *coeli ennarant deum* “the skies describe God”. For Bhāskara the earth and other elements of the universe are bodies of Śiva. All that is intense poetry of the most elevated level. And that could occur only in an astronomical account.

A description of seasons is a must for a Sanskrit poet. Bhāskara is not an exception to the rule. He wanted his *Siddhāntaśiromaṇi* to be a *kāvya*. He has done it by describing the six seasons in a section of eleven stanzas, two for *vasanta* “spring”, two for *grīṣma* “summer”, three for *varṣa* “rains”, one for *śarat* “autumn”, one for *hemanta* “early winter”, two for *śiśira* “close of winter”. He has confessed that here under the pretext of describing the seasons he wanted simply to show that he was a poet, able to please the *rasikas*, for the

pleasure of the learned. Then he has added three stanzas in praise of poetry. This section of fifteen stanzas of poetry is a parenthesis in his scientific work. In fact the originality of Bhāskara as a poet is more in achieving a fusion of science and poetry, than in composing a pure poem.

At the close of this rapid survey the question "is poetical inspiration at the origin of some scientific ideas?" comes unfailingly to mind. There is no definite answer to such a question. We just note that poetry in Bhāskara's work is more than a varnish ornamenting the scientific exposition. It is fused with it. We have seen it in the description of the earth, in the case of the division by zero, in the appropriateness of the poetic constructions to the structures of mathematical problems in the examples of the *Līlāvatī*.

The lineaments of the face reveal the internal nature of the personality of an individual. The poetical face of Bhāskara's work betrays his inward response to the beauty of mathematical procedures and to the rationale of the universe structure.

Part II

The Līlāvatī

प्रीतिं भक्तजनस्य यो जनयते विघ्नं विनिघ्नन् स्मृतः
तं वृन्दारकवृन्दवन्दितपदं नत्वा मतङ्गाननम् ।
पाटीं सद्गणितस्य वच्मि चतुरप्रीतिप्रदां प्रस्फुटां
सङ्क्षिप्ताक्षरकोमलामलपदैर्लालित्यलीलावतीम् ॥

prītiṃ bhaktajanasya yo janayate vighnaṃ vinighnan smṛtaḥ
taṃ vṛndārakavṛndavanditapadaṃ natvā mataṅgānanam |
pāṭīṃ sadgaṇitasya vacmi caturaprītipradāṃ prasphuṭāṃ
saṅkṣiptākṣarakomalāmalapadair lālityalīlāvatīm ||

Having bowed to the elephant-headed God [Gaṇeśa]–whose feet are venerated by Gods, and who bestows happiness on devotees by way of destroying [their] obstacles when meditated upon—I expound the procedures of good mathematics in words that are concise (*saṅkṣiptākṣara*), lucid (*komala*), and flawless (*amalapada*), that would be a source of delight to the experts, absolutely unambiguous and endowed with playful elegance (*lālityalīlāvatīm*).

The *līlā* of the *Līlāvatī*

K. Ramasubramanian, K. Mahesh, Aditya Kolachana*

1 Introduction

Sanskrit literature is rich in a variety of texts across a wide range of disciplines including mathematics, astronomy, medicine, history, philosophy, drama, mythology and so on. A perusal of literature around the world will show that, while it is common to compose works related to history, literature, mythology etc. in metrical form, scientific literature—due to its use of technical vocabulary, as well as demands of brevity, clarity and precision—is usually written in prose. However, in contrast to this general trend, a surprisingly large corpus of scientific literature in Sanskrit is composed in the form of verses, including works in mathematics, astronomy, medicine, linguistics etc.

The *Līlāvatī* is one such treatise dealing with elementary arithmetic and geometry, which is widely renowned for its clear elucidation of rules, and demonstration of their application in daily life through brilliant examples. In the opening verse of *Līlāvatī*, having offered his veneration to Lord Gaṇeśa, Bhāskara promises the readers that he is going to compose a work that is endowed with playful elegance (*lālitya-līlāvatī*), and which will also be a source of delight to the experts (*caturaprītipradā*). What is this playful elegance (*lālitya-līlā*) that Bhāskara is referring to which has made the work so popular and acclaimed—for almost a millennium since its composition in twelfth century—is the question we attempt to answer in the present paper.

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Though a large number of commentaries and translations of this text are available in several languages,[1] all these works mostly tend to focus on the mathematical aspects of the text. They do not attempt to investigate whether Bhāskara is justified in making the above claims, nor do they highlight the reasons for *Līlāvatī*'s enduring charm (*līlā*). Our aim in the paper is to look into this aspect of the work. To this end, and to ensure an objective study, we highlight a few technical and literary features that add elegance and charm to any text in Section 2. In Section 3, we discuss the merits of *Līlāvatī* from a technical standpoint. Next, as a precursor to discussing the literary merits of *Līlāvatī*, we give a brief introduction to *alaṅkāras* in Section 4, and discuss the specific merits of the text in Section 5. In Section 6, after giving a very brief overview of the contents of the *Līlāvatī*, we discuss some of the unique contributions made by Bhāskara in the *Līlāvatī* from a mathematical standpoint. Finally, we summarise our discussion in Section 7.

2 Features that contribute to the elegance and charm of literature

Human imagination knows no bounds, and humanity has produced a variety of literature in a diverse range of topics and languages. Different cultures perhaps have different views about what contributes to the elegance and charm of literature, and these too may vary depending upon the topic and nature of the text. However, a perusal of world literature will show that even within this bewildering assortment of tastes and styles spanning countries and cultures, some of the highly regarded works have certain aspects in common that include:

1. *saṅkṣipta-prasphuṭa-kathanam*: succinct and clear presentation, and
2. *vicitra-udāharaṇa-citraṇam*: use of captivating examples drawn from various quarters to effectively support the narrative.

Additionally, if the text is composed in verse form, the poet (and thereby the work) is evaluated on

3. the capacity to blend the work with variety of enchanting poetic metres (*nānā-chandaḥ-prayogaḥ*), and choose the one that would suit the most in a given context,

[1] For instance see Gaṇeśa's *Buddhivilāsinī* [Līlā1937], Śaṅkara's *Kriyākramakarī* [Līlā1975], [Col1817], or [Tay1816].

4. the use of clever witticisms and beautiful phrases (*ramaṇīya-padavinyāsaḥ*), and
5. the perspicacious ability to employ appropriate poetic flourishes (*alaṅkāras*).

Whereas the satisfaction of the first two criteria contributes towards the technical merit of a text, the satisfaction of the latter three contributes towards its literary merits. Based on the perusal of literature, it is evident that works that satisfy most of the above criteria can enjoy lasting popularity among lay persons and scholars alike.

In the following sections, we shall attempt to show how *Līlāvatī* satisfies most of the above criteria.

3 The *līlā* of *Līlāvatī* from a technical standpoint

In this section, we briefly review the *Līlāvatī* from the technical standpoints of (i) brevity, clarity, and precision, and (ii) use of captivating examples. The *līlā* of *Līlāvatī* is best brought out when compared with some other well known mathematical works. Most works of mathematics, composed both before and after *Līlāvatī*, do not satisfy all of the above criteria. For instance, Āryabhaṭa's famous treatise *Āryabhaṭīya*—though contributing greatly towards the development of mathematics and astronomy—is somewhat terse in some places, and does not present any examples. Brahmagupta's *Brāhmasphuṭasiddhānta* too essentially follows the style of *Āryabhaṭīya*. In addition, these texts are composed entirely in the drab *āryā* metre, and hardly employ any poetic flourishes.[2]

A few other works in mathematics such as *Gaṇitatilaka* of Śrīpati, *Gaṇitasārasaṅgraha* of Mahāvīrācārya, and *Gaṇitakaumudī* of Nārāyaṇa Paṇḍita do employ a variety of metres and occasionally use poetic flourishes. Though these works fulfil some of the criteria listed in the previous section, they do fall short in one or the other respect. In contrast to the above works, *Līlāvatī* maintains a harmonious balance between brevity and clarity, and makes use of a large number of enchanting examples besides employing a variety of poetic flourishes. We illustrate this with a few examples in the following sections.

[2] We mention this only to contrast *Līlāvatī* with these texts, and not to undermine the importance of them (*nahi nindānyāya*).

3.1 Example to illustrate saṅkṣipta-prasphuṭakathanam

Here, we give an example of how Bhāskara harmonises the conflicting goals of brevity and clarity in the *Līlāvatī*. The following verse is from the chapter on arithmetic and geometric progressions:[3]

सैकपदघ्नपदार्धमथैका-
द्यङ्कयुतिः किल सङ्कलिताख्या ।
सा द्वियुतेन पदेन विनिघ्नी
स्यात्त्रिहृता खलु सङ्कलितैक्यम् ॥११७॥ । दोधकम् ।

saikapadaghnapadārdhamathaikā-
dyaṅkayutiḥ kila saṅkalitākhyā |
sā dviyutena padena vinighnī
syāttrihṛtā khalu saṅkalitaikyam ||117||

Now, the sum of the numbers starting with one, called *saṅkalita*, is indeed half the number of terms (*pada*) multiplied by the *pada* added by one. That [*saṅkalita*] multiplied by the *pada* [which is] added by two, [and] divided by three would indeed be the sum of sums (*saṅkalitaikya*).

This verse gives the relations for (i) *saṅkalita*: the summation of the first n integers starting with the number one, and (ii) *saṅkalitaikya*: the sum of sums.

$$\textit{saṅkalita} \qquad 1+2+3+4+\ldots+n = \frac{n(n+1)}{2} \tag{1}$$

$$\textit{saṅkalitaikya} \qquad 1+(1+2)+\ldots+\frac{n(n+1)}{2} = \frac{n(n+1)(n+2)}{1.2.3}. \tag{2}$$

The brevity, clarity and charm with which Bhāskara has been able to present the above results can be readily appreciated by comparing it with the following verse of Brahmagupta which essentially presents the second of the above relations [BSS1902, ch. 12.19, p. 284] :

एकोत्तरमेकाद्यं यदिष्टगच्छस्य[4] भवति सङ्कलितम् ।
तद्द्वियुतगच्छगुणितं त्रिहृतं सङ्कलितसङ्कलितम् ॥ । आर्या ।

ekottaramekādyaṃ yadiṣṭagacchasya bhavati saṅkalitam |
taddviyutagacchaguṇitaṃ trihṛtaṃ saṅkalitasaṅkalitam ||

If the sum (*saṅkalita*) [of a progression] of given number of terms (*gaccha*) has one as first term and increment, then that [*saṅkalita*] multiplied by the number of terms added by two, [and] divided by three is the sum of sums (*saṅkalita-saṅkalita*).

[3] Here, the verses quoted from Līlāvatī are numbered as given in [Līlā1937].

[4] यदीष्टगच्छस्य in [BSS1902]. Given reading, which is as per India Office Library manuscript, Eggeling 2769, is better as the तत् in the third quarter of the verse gets its correlative pronoun यत्.

In comparison to the clarity of Bhāskara's verse where the two relations described are readily understood, deciphering Brahmagupta's verse—which presents only (2)—requires both knowledge of the context, as well as effort. Moreover, Bhāskara has composed his verse in the appealing *dodhaka* metre, while Brahmagupta—as stated before—has employed the somewhat bland *āryā* metre. Therefore, though both the verses essentially give the same formula, Bhāskara's verse is enjoyable, while Brahmagupta's verse is not that inspiring.

In short, what we would like to convey here is that the verses in the *Līlāvatī* are generally characterized by a beautiful blend of three desirable qualities of a text, namely:

1. संक्षिप्तता – the quality of being brief
2. प्रस्फुटता – the quality of being clear
3. लालित्य – the quality of being elegant.

In a text like the *Līlāvatī*, where various mathematical rules and formulae have to be enunciated, blending the first two into a verse may not be that difficult, whereas to weave in the third with the others is not easy. However, Bhāskara seems to have succeeded in that as well. Elegance can be brought into the text by a variety of techniques such as

1. composing verses in metres that have beautiful rhythms
2. employing poetic flourishes like alliteration, anaphora and epiphora
3. employing apt similes and nice puns
4. constructing brilliant examples.

While it may be easier to incorporate all the above features while constructing illustrative examples, it would indeed be challenging to employ them when it comes to describing mathematical rules. What is remarkable is, Bhāskara does succeed in bringing in elegance even in mathematical rules by employing at least the first two of the above features, as seen in the above example. This is the *līlā* of the *Līlāvatī*.

3.2 Examples of vicitra-udāharaṇa-citraṇam

Here, we show Bhāskara's ability to construct very imaginative examples that are at once captivating and elevating.

Example 1

The following verse from the chapter on geometry poses a problem involving right-angled triangles:

यदि समभुवि वेणुः द्वित्रिपाणिप्रमाणो
गणक पवनवेगात् एकदेशे स भग्नः ।
भुवि नृपमितहस्तेष्वङ्ग लग्नं तदग्रं
कथय कतिषु मूलादेष भग्नः करेषु ॥१५०॥ । मालिनी ।

yadi samabhuvi veṇuḥ dvitripāṇipramāṇo
gaṇaka pavanavegāt ekadeśe sa bhagnaḥ |
bhuvi nṛpamitahasteṣvaṅga lagnaṃ tadagraṃ
kathaya katiṣu mūlādeṣa bhagnaḥ kareṣu ||150||

O mathematician! If a bamboo measuring thirty-two *pāṇis*[5] (lit. hands) on a level ground (*sama-bhuvi*) was broken at one place due to wind speed (*pavana-vega*), [and] its tip touched the ground at a distance of sixteen *hastas*, [then] O dear![6] State how many *karas* (*hastas*) from the root this [bamboo] is broken at.

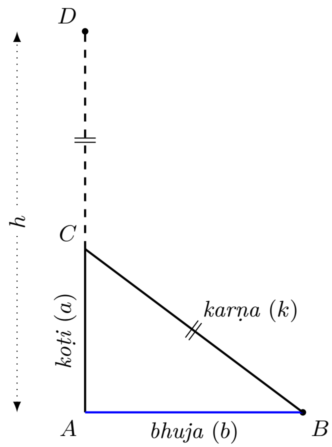

Figure 1: The bamboo example borrowed by Bhāskara from Bhāskara I.

The verse describes a bamboo of height (h) equalling 32 *hastas*, which is broken at an unknown height due to the wind as shown in Figure 1. The broken (but still attached) portion of the bamboo falls to the ground such that its tip is at a distance of 16 *hastas* from the foot of the bamboo. We need

[5] A unit of linear measure, same as *hasta* or *kara*.

[6] Here, the word *aṅga* is used is its colloquial sense as an addressing word, and refers to a person as dear as an inseparable part of the body.

to determine the height at which the bamboo is broken. Using a rule stated earlier by Bhāskara, the height (*koṭi*) at which the bamboo is broken can be determined as follows:

$$\textit{koṭi} = \frac{1}{2} \times \left(32 - \frac{16^2}{32}\right) = 12 \ \textit{hastas}.$$

We can then also determine the hypotenuse (*karṇa*), which is equal to the length of the upper portion of the bamboo as follows:

$$\textit{karṇa} = 32 - 12 = 20 \ \textit{hastas}.$$

This verse makes use of an example from nature to help the students readily visualise what may otherwise seem a difficult and abstract mathematical problem. This not only helps the student to understand the situation, but also motivates by presenting it as a real-life problem. This example, with only the numbers changed proportionally, is actually borrowed from Bhāskara I's *Āryabhaṭīya-bhāṣya* [AB1976, p. 100]:

> षोडशहस्तो वंशः पवनेन निपातितः स्वमूलात् ।
> अष्टौ गत्वा पतितः कस्मिन् भग्नो मरुत्वतो वाच्यः ॥ । आर्या ।
>
> *ṣoḍaśahasto vaṃśaḥ pavanena nipātitaḥ svamūlāt |*
> *aṣṭau gatvā patitaḥ kasmin bhagno marutvato vācyaḥ ||*
>
> A bamboo of sixteen *hastas* was made to fall by the wind. It fell such that its tip hit the ground at eight *hastas* from the root. Where was it broken by the Lord of the wind, is to be said.

Bhāskara I has therefore partly satisfied the requirement of *vicitra-udāharaṇa-citraṇam* by beautifully weaving in a nature related observation into a mathematical work. However, Bhāskara surpasses Bhāskara I with his superior choice of words and metre. The *mālinī* employed by the author of *Līlāvatī* lends itself far better to rhythm and melody, with the result that Bhāskara's restatement of this example surpasses its earlier form. This is best experienced in the second quarter where the characteristic of the *mālinī* metre compels the fast utterance of the phrase *gaṇaka pavanavegāt*, which then appropriately gives the sense of the speed of the wind.

In comparing the verses of the two scholars, it is also worth noting the use of the phrase *sama-bhuvi* (level ground) by Bhāskara. Unless the ground is perfectly flat, the triangle resulting from the two parts of the broken bamboo and the ground would not be right-angled. Therefore, the use of this phrase greatly enhances the precision of the verse. Thus, we find Bhāskara's verse to not only be more elegant, but also more precise. This is the *līlā* of *Līlāvatī*.

Example 2

The following verse is from the chapter on quadratic equations in the *Līlāvatī*:

यातं हंसकुलस्य मूलदशकं मेघागमे मानसं
प्रोड्डीय स्थलपद्मिनीवनमगादष्टांशकोऽम्भस्तटात् ।
बाले बालमृणालशालिनि जले केलिक्रियालालसं
दृष्टं हंसयुगत्रयं च सकलां यूथस्य संख्यां वद ॥६९॥ । शार्दूलविक्रीडितम् ।

yātaṃ haṃsakulasya mūladaśakaṃ meghāgame mānasaṃ
proḍḍīya sthalapadminīvanamagādaṣṭāṃśako'mbhastaṭāt |
bāle bālamṛṇālaśālini jale kelikriyālālasaṃ
dṛṣṭaṃ haṃsayugatrayaṃ ca sakalāṃ yūthasya saṃkhyāṃ vada ||69||

Of a herd of swans (at a water body), ten times the square root went (migrated) to Mānasa (lake) on the approach of clouds (rainy season). One-eighth having flown, went to a forest of *sthalapadminī* (*Hibiscus mutabilis*) from the shore of the water. Three pairs of swans were observed to be absorbed sporting in water having delicate stalks of lotuses. O girl! State the total number [of swans] the group has.

In this verse, Bhāskara imagines a herd of swans which migrate to different regions on the approach of the rains, and constructs a problem involving quadratic equations based on this migration. He evocatively describes one group of swans which migrates to the legendary Mānasa lake (which is famous in Indian literature for swans), another group which migrates to an enticing forest of *sthalapadminī* flowers, and a laggard group of three pairs of swans still sporting in a water body graced by beautiful lotuses. We need to determine the total number of swans in the herd.

Forcing the mind towards mathematics from this enticing description, the given problem can be represented as the following quadratic equation, where x^2 is the total number of swans:

$$x^2 = \frac{x^2}{8} + 10x + 6.$$

Solving this equation, we have $x^2 = 144$.

By invoking the legendary Mānasa lake, and the revered *haṃsa* (swan), Bhāskara captures the attention of the reader. Thus, this verse not only teaches mathematics, but also educates the students about the migration of birds, and appeals to the naturalists in all of us. This is the *līlā* of the *Līlāvatī*.[7]

[7] Also, the repetition of the syllable *la* in the third quarter of the above verse is a good example of the *śabdālaṅkāra* called *vṛtti-anuprāsa*, about which we discuss in greater detail in Section 5.

Example 3

Another example that is worth citing here is a problem related to permutations that appears towards the end of the text:

पाशाङ्कुशाहिडमरूककपालशूलैः
खट्वाङ्गशक्तिशरचापयुतैर्भवन्ति ।
अन्योन्यहस्तकलितैः कति मूर्तिभेदाः
शंभोर्हरेरिव गदारिसरोजशङ्खैः ॥२६३॥ । वसन्ततिलका ।

pāśāṅkuśāhiḍamarūkakapālaśūlaiḥ
khaṭvāṅgaśaktiśaracāpayutairbhavanti |
anyonyahastakalitaiḥ kati mūrtibhedāḥ
śaṃbhorharereiva gadārisarojaśaṅkhaiḥ ||263||

Like the varieties of sculptures (*mūrtis*) of Hari (Viṣṇu) with a mace, a disc, a lotus, and a conch, how many varieties of *mūrtis* of Śambhu (Śiva) are possible by interchangeably placing a rope, a hook, a snake, a *ḍamarū*, a skull, the *śūla*, the *khaṭvāṅga*, a spear, an arrow, and a bow in his [ten] hands?[8]

Here, Bhāskara presents a problem involving permutations by borrowing certain descriptions from India's sacred literature. There, the god Viṣṇu is described as holding a mace, a disc, a lotus, and a conch with his four hands.[9] This image is widely prevalent not only in literature, but in sculpture, as well as paintings. However, it is possible to find variations among the *mūrtis* and paintings with the positions of the different objects interchanged between the four hands.

Similarly, Śiva has been described in literature as having five faces, ten hands, and holding a variety of weapons.[10] Just like it is possible to create

[8] The snake here probably refers to *Vāsuki*, the king of the serpents, which is usually depicted as coiled around Śiva's neck. The *ḍamarū* is Śiva's drum, the sound produced from which is said to have inspired Pāṇini to write the famous *Māheśvara-sūtras*, which form the basis of Sanskrit grammar. *Śūla* refers to Śiva's famous weapon—the *Triśūla*, which is a three-pronged spear. *Khaṭvāṅga* is another weapon of Śiva—a club with a skull at the top. The bow referred to here would be Śiva's famous bow *Pināka*, which plays an important role early in the *Rāmāyaṇa*, during the *svayaṃvara* of Sītā.

[9] Viṣṇu's mace is called *kaumodakī*. His disc is the famous *sudarṣana-chakra*, and his conch is known as *pāñcajanya*.

[10] The following famous verse describes Śiva slightly differently from Bhāskara:

शान्तं पद्मासनस्थं शशधरमकुटं पञ्चवक्त्रं त्रिनेत्रं
शूलं वज्रं च खड्गं परशुमभयदं दक्षभागे वहन्तम् ।
नागं पाशं च घण्टां प्रलयहुतवहं साङ्कुशं वामभागे
नानालङ्कारयुक्तं स्फटिकमणिनिभं पार्वतीशं नमामि ॥

śāntaṃ padmāsanasthaṃ śaśadharamakuṭaṃ pañcavaktraṃ trinetraṃ
śūlaṃ vajraṃ ca khaḍgaṃ paraśumabhayadaṃ dakṣabhāge vahantam |

different *mūrtis* by interchanging the items held in Viṣṇu's hands, this verse challenges the students to determine the possible number of *mūrtis* which can be obtained by interchanging the ten items in Śiva's hands. Solving, we determine that there are $10! = 3628800$ possible *mūrtis* of Śiva, and $4! = 24$ possible *mūrtis* of Hari.

It can be seen here that Bhāskara once again effectively relates the mathematics being taught in the classroom to the texts and imagery the students are frequently exposed to, which makes the process of learning more meaningful, delightful, and effective.[11]

Example 4

The following verse cleverly makes use of a social custom (*sāmājika-vṛttam*) to teach arithmetic progressions.

आद्ये दिने द्रम्मचतुष्टयं यो दत्त्वा द्विजेभ्योऽनुदिनं प्रवृत्तः ।
दातुं सखे पञ्चचयेन पक्षे द्रम्मा वद द्राक्कति तेन दत्ताः ॥१२२॥ । इन्द्रवज्रा ।

ādye dine drammacatuṣṭayaṃ yo dattvā dvijebhyo'nudinaṃ pravṛttaḥ |
dātuṃ sakhe pañcacayena pakṣe drammā vada drākkati tena dattāḥ ||122||

A person who having donated four *drammas* to *dvijas* (brahmins) on the first day (in charity), continued to give by an increment of five on the following days. O friend! Quickly tell how many *drammas* were given by him in a fortnight.

A person gives 4 *drammas* in charity on the first day, and increases this by an amount of 5 *drammas* on each subsequent day. Given that he gives to charity for a total of 15 days, we need to determine the total sum given by that person in charity.

We are given the first term of the progression $a_1 = 4$, common difference $d = 5$, and number of terms $n = 15$. Using various formulae given in *Līlāvatī*, we first determine the last term, then the middle term, and finally the sum of the progression as follows:

---

nāgaṃ pāśaṃ ca ghaṇṭāṃ pralayahutavahaṃ sāṅkuśaṃ vāmabhāge
nānālaṅkārayuktaṃ sphaṭikamaṇinibhaṃ pārvatīśaṃ namāmi ||
I bow to the Lord (husband) of Pārvatī, adorned with many ornaments, resplendent like a crystal, and the embodiment of peace, who is seated in the *padmāsana* (lotus pose), wears the Moon as a crown, has five faces and three eyes, and carries the *śūla*, *vajra*, a sword, an axe, and the *abhaya-mudrā* on the right side, and a snake, a rope, a bell, the *pralayāgni*, and a hook on the left side.

[11] For an interesting discussion on the iconography described in this verse, see [Sar2006].

$$\begin{aligned} \text{Last term of the progression} \qquad & a_n = 4 + (15-1) \times 5 = 74 \\ \text{Middle term of the progression} \qquad & a_m = \frac{4+74}{2} = 39 \\ \text{Sum of the progression} \qquad & S_A = 39 \times 15 = 585 \ \textit{drammas.} \end{aligned}$$

Through the above examples we have tried to show how Bhāskara balanced the competing claims of brevity and clarity, and how he made use of ingenious examples for effective transmission of mathematical principles.

Now, we move on to discuss how Bhāskara makes use of a number of poetic embellishments, collectively known as *alaṅkāras*, to elevate the beauty of the verses of *Līlāvatī* to a sublime level, achieved only rarely by poets like Kālidāsa. However, as one first needs to understand the basic theory behind *alaṅkāras* to appreciate Bhāskara's poetic genius, we briefly describe the concept of *alaṅkāras* in Sanskrit literature in the following section.

4 *Alaṅkāras* in Sanskrit literature

The term *alaṅkāra* literally means 'ornament' or 'decoration'. However in the context of Sanskrit literature it refers to a variety of figures of speech or poetic flourishes that instantly creates a great sense of admiration and can also evoke *rasa* in the reader. It also adds to the charm and elegance of the literature that is being composed. At the beginning of the fifth chapter (*mayūkha*) of his work *Candrāloka*, Jayadeva presents the following definition of an *alaṅkāra* [CaLo1950, pp. 91–92]:

शब्दार्थयोः प्रसिद्ध्या वा कवेः प्रौढिवशेन वा ।
हारादिवदलङ्कारः सन्निवेशो मनोहरः ॥११७॥

śabdārthayoḥ prasiddhyā vā kaveḥ prauḍhivaśena vā |
hārādivadalaṅkāraḥ sanniveśo manoharaḥ ||117||

The exquisite beauty that is brought in either by the adept use of words and meaning, or by brilliant poetic imagination, which becomes an ornament [of a poem], just like garland etc., is [called] *alaṅkāra*.

To illustrate the point as it were, consider the following example. Let's suppose a teacher is walking along with three students in the evening twilight. Having noticed a serpent on the branch of a shrivelled tree in front of him, he asks his three disciples to describe the event. They come up with the following descriptions:

A : पुरतः शुष्के वृक्षे शाखायामग्रे कश्चित् सर्पोऽस्तीति भाति ।

purataḥ śuṣke vṛkṣe śākhāyāmagre kaścit sarpo'stīti bhāti |

B : वृक्षे शुष्के त्वग्रे पुरतः मन्ये पन्नगराजोऽप्यास्ते ।

vṛkṣe śuṣke tvagre purataḥ manye pannagarājo'pyāste |

C : नीरसतरुरिव निवसति पुरतः फणिमणिरञ्जितमञ्जुलशाखः ।

nīrasataruriva nivasati purataḥ phaṇimaṇirañjitamañjulaśākhaḥ |

All the three descriptions given above essentially narrate the same event. But the impressions they leave behind in the minds of the reader are very different. Description A is quite bland and insipid, and besides serving the purpose of conveying the required information, doesn't generate any joy in the reader. Description B, though employing many of the same words as A, has a certain elegance and beauty to it. Whereas, description C is strikingly beautiful and instantly delights the minds of the readers. What makes these descriptions very different from one another?

It is the adept use of words in a specific sequence—generally called *śabdavinyāsa* or *padavinyāsa*—that works wonders. Just as the style of attire, and the use of ornaments, can completely change the look of an individual (the core person remaining the same), so too can the choice of the words (*padas*) and their arrangement (*vinyāsa*) create utterly different impressions in the minds of the readers.

Given the importance of the use of *alaṅkāras* to add charm to literature, an entire discipline of study (*alaṅkāraśāstra*) has been systematically developed in India describing and classifying various *alaṅkāras*. A long line of scholars starting with Bharata, and including Bhāmaha, Daṇḍin, Vāmana, Udbhaṭa, Rudraṭa, Ānandavardhana, Abhinavagupta, Kuntaka, Bhoja, Mammaṭa, Ruyyaka, Vidyānātha, Viśvanātha, and Jagannātha have made numerous contributions to the growth and development of *alaṅkāraśāstra* by identifying new *alaṅkāras*, and classifying them into appropriate categories as they deemed suitable. While Bharata described only four *alaṅkāras*, subsequent scholars have expanded this list to as many as two hundred and more.

The *alaṅkāras* are broadly divided into the two categories, namely: *śabdālaṅkāra* and *arthālaṅkāra*. *Śabdālaṅkāras* are poetic embellishments having to do with the recurrence of phonetic features that bring joy to the reader, even as the text is being read without getting into its meaning. It is indeed the use of *śabdālaṅkāras* which is the cause of the charm and beauty in descriptions B and C above. *Arthālaṅkāras* on the other hand have to do with the meaning of the words and different phrases employed in the verse. In the next section we will see the use of various forms of these *alaṅkāras* in the *Līlāvatī*.

5 Use of *alaṅkāras* in *Līlāvatī*

The *Līlāvatī* is a work of 272 verses, which has been composed using twenty-three poetic metres[12] and numerous *alaṅkāras*. Perhaps as a tribute to the display of this wide poetic repertoire, Bhāskara's grandson Caṅgadeva alludes to the poetic genius of the author of *Līlāvatī* with the following phrase in an inscription eulogising his grandfather:[13]

> स्वच्छन्दो यश्छन्दसि
>
> *svacchando yaśchandasi*
>
> At one's own will, the one who can employ [a vivid variety of] poetic metres.

Anyone who reads Bhāskara's *Līlāvatī* would be easily convinced that the above description has a basis in fact, and is certainly not due to mere familial pride.

In what follows, we discuss a few select verses of *Līlāvatī* which highlight the poetic genius of Bhāskara. The section has been divided into two parts to showcase a few examples each of the use of *śabdālaṅkāra* and *arthālaṅkāra* in the text respectively. Some of the examples below also demonstrate the use of multiple *alaṅkāras*, in the same verse.

5.1 Examples of śabdālaṅkāras

Though there are several kinds of *śabdālaṅkāras*, they can be broadly divided into two types, *anuprāsa* and *yamaka*, without entering into the technicalities of the classification schemes used by different scholars. The former is generally concerned only with the alliteration and consonance of syllables, whereas, the latter has to do with the meanings of the repeated phrases or words as well. We will now demonstrate the use of different kinds of *anuprāsa* and *yamaka* in the *Līlāvatī* through some illustrative examples.

Example 1

The *Līlāvatī* commences with the following invocatory verse, wherein Bhāskara having offered his venerations to Gaṇeśa for the successful completion of the intended work, clearly spells out the scope of the work.

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[12] See Table 1.

[13] See [Daj1865]. The inscription also notes the proficiency of Bhāskara in several branches of *śāstra*, as well as the respect that he enjoyed among the scholarly community.

Table 1: Metres used in the *Līlāvatī*.

No.	Metre	Type	Syllables per quarter	Number of verses
1	उपजातिः	गणवृत्तम्	11	62
2	अनुष्टुभ्	गणवृत्तम्	8	58
3	इन्द्रवज्रा	गणवृत्तम्	11	33
4	वसन्ततिलका	गणवृत्तम्	14	26
5	आर्या	मात्रावृत्तम्	-	25
6	शार्दूलविक्रीडितम्	गणवृत्तम्	19	21
7	रथोद्धता	गणवृत्तम्	11	8
8	मालिनी	गणवृत्तम्	15	6
9	दोधकम्	गणवृत्तम्	11	5
10	उपेन्द्रवज्रा	गणवृत्तम्	11	4
11	आख्यानिकी	गणवृत्तम्	11	3
12	स्रग्धरा	गणवृत्तम्	21	3
13	गीतिः	मात्रावृत्तम्	-	2
14	द्रुतविलम्बितम्	गणवृत्तम्	12	2
15	भुजङ्गप्रयातम्	गणवृत्तम्	12	2
16	मन्दाक्रान्ता	गणवृत्तम्	17	2
17	वैतालीयम्	गणवृत्तम्	10/11	2
18	शालिनी	गणवृत्तम्	11	2
19	स्रग्विणी	गणवृत्तम्	12	2
20	विपरीताख्यानिकी	गणवृत्तम्	11	1
21	वंशस्थम्	गणवृत्तम्	12	1
22	शिखरिणी	गणवृत्तम्	17	1
23	स्वागता	गणवृत्तम्	11	1
		Total		272

प्रीतिं भक्तजनस्य यो जनयते विघ्नं विनिघ्नन् स्मृत-
स्तं वृन्दारकवृन्दवन्दितपदं नत्वा मतङ्गाननम् ।
पाटीं सद्गणितस्य वच्मि चतुरप्रीतिप्रदां प्रस्फुटां
सङ्क्षिप्ताक्षरकोमलामलपदैर्लालित्यलीलावतीम् ॥१॥ । शार्दूलविक्रीडितम् ।

prītiṃ bhaktajanasya yo janayate vighnaṃ vinighnan smṛta-
staṃ vṛndārakavṛndavanditapadaṃ natvā mataṅgānanam |
pāṭīṃ sadgaṇitasya vacmi caturaprītipradāṃ prasphuṭāṃ
saṅkṣiptākṣarakomalāmalapadairlālityalīlāvatīm ||1||

Having bowed (worshipped) to the elephant-headed god (Gaṇeśa)—whose feet are venerated by gods, and who bestows happiness on devotees by way of destroying [their] obstacles when remembered (meditated upon)—I expound the procedures of good mathematics in words that are concise, lucid, and flawless, that would be a source of delight to the experts, absolutely unambiguous and endowed with playful elegance.

Bhāskara packs this verse with a variety of *śabdālaṅkāras* and *arthālaṅkāras*, giving an indication of what to expect in the rest of the text. These are discussed below.

***Vṛtti-anuprāsa*:** This *alaṅkāra* is said to be present wherever one finds repeated occurrence of vowels or consonants. The occurrence can be of an isolated consonant or a group of consonants. Jayadeva defines *vṛtti-anuprāsa* as [CaLo1950, p. 95]:

आवृत्तवर्णसम्पूर्णं वृत्त्यनुप्रासवद्वचः ।
āvṛttavarṇasampūrṇaṃ vṛttyanuprāsavadvacaḥ |
The speech that is filled with repeated syllables possesses *vṛtti-anuprāsa*.

The above invocatory verse forms a good example of *vṛtti-anuprāsa* as certain syllables in different quarters get repeated several times as shown in Table 2.

Table 2: *Vṛtti-anuprāsa* in the invocatory verse of the *Līlāvatī*.

Quarter	Phrase	Repeated syllables
2	वृन्दारकवृन्दवन्दितपदं	व, द
3	प्रीतिप्रदां प्रस्फुटां	प
4	कोमलामलपदैर्लालित्यलीलावतीम्	ल

Moreover, it is observed that the syllable *na* appears frequently in the first two quarters, while *ta* appears throughout the verse. All this amply demonstrates the use of *vṛtti-anuprāsa*.

***Cheka-anuprāsa*:** This *alaṅkāra* is said to be present when a group of vowels and consonants have a single repetition. *Sāhityadarpaṇa* states [SāDa1875, ch. 10.2]:

छेको व्यञ्जनसङ्घस्य सकृत्साम्यमनेकधा ।
cheko vyañjanasaṅghasya sakṛtsāmyamanekadhā|

One time ordered repetition of the same group of [vowels and] consonants is *cheka*.

The above invocatory verse constitutes a good example of this *anuprāsa* as we find several phrases (*jana*, *vṛnda*, *mala*) occurring exactly twice as shown in Table 3.

Table 3: *Cheka-anuprāsa* in the invocatory verse of the *Līlāvatī*.

Quarter	Phrase	Repeated phrases
1	भक्तजनस्य यो जनयते	जन
2	वृन्दारकवृन्दवन्दितपदं	वृन्द
4	कोमलामल	मल

***Yamaka*:** This *alaṅkāra* is said to occur when different instances of a repeated group of vowels and consonants have different meanings. Jayadeva defines *yamaka* as [CaLo1950, pp. 98–99]:

आवृत्तवर्णस्तबकं स्तवकन्दाङ्कुरं कवेः ।
यमकं प्रथमा धुर्यमाधुर्यवचसो विदुः ॥
āvṛttavarṇastabakaṃ stavakandāṅkuraṃ kaveḥ |
yamakaṃ prathamā dhuryamādhuryavacaso viduḥ ||

The eminent poets of past who have mastered the technique of using pleasant words consider *yamaka* as [an instance of] repeated use of a group of syllables. They also consider it as the sprout of the bulbous root of praise.

It may be noted that, in the first phrase in Table 3, *jana* is used in the sense of 'person' (*bhaktajana*) and 'generate' (*janayate*) in the first and second instances respectively. Similarly, in the second phrase in the same table, *vṛnda* first appears in the term *vṛndāraka* (deity), where it is meaningless in itself. In its second appearance, it is used in the sense of 'group' (*vṛnda*). Thus, the invocatory verse cited above exhibits the use of *yamaka* also.

Example 2

The following verse is yet another invocatory verse seeking the blessings of Gaṇeśa, that is found at the beginning of the mathematics portion of the *Līlāvatī*. This verse perhaps illustrates the best use of *vṛtti-anuprāsa*:

लीलागललुलल्लोलकालव्यालविलासिने ।
गणेशाय नमो नीलकमलामलकान्तये ॥९॥ । अनुष्टुभ् ।

līlāgalalulallolakālavyālavilāsine |
gaṇeśāya namo nīlakamalāmalakāntaye ||9||

Salutations to Gaṇeśa, who is resplendent like the spotless (*amala*) blue lotus, and who is playing with a black serpent which is gracefully swaying (*lulat*), coiling and uncoiling (*lolat*) around the neck.

Here, it is observed that *la* is repeated eleven times in the first half of the verse and three times in the second half, thus forming a classic example of the *vṛtti-anuprāsa*. The imagery of Gaṇeśa playing with a snake further enhances the beauty of the verse. The *arthālaṅkāras* present here are discussed in a later section.

Example 3

The verse below which deals with the fundamental arithmetic operations forms a wonderful example of *cheka-anuprāsa*:

अये बाले लीलावति मतिमति ब्रूहि सहितान्
द्विपञ्चद्वात्रिंशत्त्रिनवतिशताष्टादश दश ।
शतोपेतानेतानयुतवियुतांश्चापि वद मे
यदि व्यक्ते युक्तिव्यवकलनमार्गेऽसि कुशला ॥१३॥ । शिखरिणी ।

aye bāle līlāvati matimati brūhi sahitān
dvipañcadvātriṃśattrinavatiśatāṣṭādaśa daśa |
śatopetānetānayutaviyutāṃścāpi vada me
yadi vyakte yuktivyavakalanamārge'si kuśalā ||13||

O intelligent girl Līlāvati! Tell me [the sum when] two, five, thirty two, one hundred and ninety three, eighteen, and ten, are put together with hundred. Also tell me [the result of] those subtracted from ten thousand, if you are skilled in arithmetic methods (*vyakte*) of addition (*yukti*) and subtraction (*vyavakalana*).

For a verse to exhibit *cheka-anuprāsa*, some phrases must get repeated once. In this verse we find three such instances—the phrases *mati*, *daśa*, and *yuta* get repeated in the first, second, and third quarters respectively. All these are instances of single repetition and hence this is a good example of the use of *cheka-anuprāsa.*

Example 4

In the context of describing the procedures for finding cubes and cube roots of numbers, Bhāskara gives the following verse as an illustrative example, making telling use of the *vṛtti-anuprāsa*:

नवघनं त्रिघनस्य घनं तथा
कथय पञ्चघनस्य घनं च मे ।
घनपदं च ततोऽपि घनात् सखे
यदि घनेऽस्ति घना भवतो मतिः ॥२७॥ । द्रुतविलम्बितम् ।

navaghanaṃ trighanasya ghanaṃ tathā
kathaya pañcaghanasya ghanaṃ ca me |
ghanapadaṃ ca tato'pi ghanāt sakhe
yadi ghane'sti ghanā bhavato matiḥ ||27||

Friend, tell me nine cubed, the cube of three cubed, and similarly, the cube of five cubed. And then also [tell] the cube root from the cubes [obtained] if your intellect is strong in [the calculation of] cubes.

It may be noted that the phrase *ghana* is repeated nine times in this verse, sometimes in its isolated form and sometimes as a part of another compound term. This repetition combined with the use of the delightful *drutavilambita* metre contributes towards a very beautiful verse.

Example 5

Yamaka is employed in the following verse of the *Līlāvatī* which poses a problem involving a quadratic equation:

बाले मरालकुलमूलदलानि सप्त
तीरे विलासभरमन्थरगाण्यपश्यम् ।
कुर्वच्च केलिकलहं कलहंसयुग्मं
शेषं जले वद मरालकुलप्रमाणम् ॥६७॥ । वसन्ततिलका ।

bāle marālakulamūladalāni sapta
tīre vilāsabharamantharagāṇyapaśyam |
kurvacca kelikalahaṃ kalahaṃsayugmaṃ
śeṣaṃ jale vada marālakulapramāṇam ||67||

O girl! I saw seven halves (seven times half) of the square root of a gaggle of geese sauntering jovially on the shore, and a remaining couple play-fighting in the water. State the measure of the gaggle of geese.

The mathematical equation that is posed by the verse is:

$$x^2 - \frac{7}{2}x = 2.$$

Setting aside the mathematical aspect, from a purely literary view point the verse serves as a good illustration of the use of *yamaka*. It may be noted that in the third quarter, the same phrase *kalahaṃ* occurs twice. In the first instance, *kalahaṃ* is an isolated word and refers to the 'quarrel' between the geese. In the second instance, the phrase *kalahaṃ* is actually a part of the compound *kalahaṃsa-yugmaṃ*[14] (a pair of geese), where the phrase *kalahaṃ* on its own has no meaning. Recalling that *yamaka* requires the phrase as such to be meaningful only in one instance, one can conclude that the above verse constitutes a good example of the use of *yamaka*.

Example 6

Yamaka is also observed in the last quarter of the following verse, that appears in the chapter dealing with arithmetic and geometric progressions. As can be noticed, the phrase *vada* is repeated twice here. In the first instance this term is part of the word *vadana* (first term of a progression), and has no meaning on its own. In the second instance, *vada* is an isolated word which means 'state'.

पञ्चाधिकं शतं श्रेढीफलं सप्त पदं किल ।
चयं त्रयं वयं विद्मो वदनं वद नन्दन ॥१२५॥ । अनुष्टुभ् ।

pañcādhikaṃ śataṃ śreḍhīphalaṃ sapta padaṃ kila |
cayaṃ trayaṃ vayaṃ vidmo vadanaṃ vada nandana ||125||

We know that the sum of the progression (*śreḍhīphala*) is one hundred and five, number of terms (*pada*) is seven, and increment (*caya*) is three. O my dear son (*nandana*)! State the first term (*vadana*).

The *yamaka* here is also evident when we note the repetition of the three syllabled word *vadana*. In the first instance, the word *vadanaṃ* is observed to be the first word in the fourth quarter. In the second instance, if we read the last two words of the verse together, i.e. *vada* and *nandana*, then by virtue of the combination of the first syllable *na* of the last word with *vada*, we once again obtain *vadana*.

5.2 Use of arthālaṅkāras

In this section, we show the use of various *arthālaṅkāras* in the *Līlāvatī*. In scientific treatises, the use of *arthālaṅkāras* is more difficult than the use of *śabdālaṅkāras*, as they may not be required or justified in most verses. However,

[14] *kalahaṃsa* refers to a species of geese.

Bhāskara seems to have utilised every opportunity to employ *arthālaṅkāras*, and thereby enhance the beauty of the work. In order to facilitate the reader to have a better appreciation, we first present the definition of the *arthālaṅkāra*, and then proceed to illustrate it with examples from the *Līlāvatī*.

5.2.1 *Upamā*

The use of appealing similes is known as *upamā-alaṅkāra* in Sanskrit. *Upamā* is pervasive in Sanskrit literature, and Bhāskara too has made extensive use of it in the *Līlāvatī*. *Candrāloka* defines the *upamā* as follows [CaLo1950, p. 102]:

उपमा यत्र सादृश्यलक्ष्मीरुल्लसति द्वयोः ।

upamā yatra sādṛśyalakṣamīrullasati dvayoḥ |

Where the wealth of similarity shines between the two [things to be compared] there is [said to be] *upamā*.

Example 1

While there are many good examples of the use of *upamā* in the *Līlāvatī*, the following verse employs it in a particularly telling fashion. This verse appears in a section called *iṣṭakarma*, which deals with the procedure for solving single-variable linear equations.

पञ्चांशोऽलिकुलात्कदम्बमगमत्त्र्यंशः शिलीन्ध्रं तयोः
विश्लेषस्त्रिगुणो मृगाक्षि कुटजं दोलायमानोऽपरः ।
कान्ते केतकमालतीपरिमलप्राप्तैककालप्रिया-
दूताहूत इतस्ततो भ्रमति खे भृङ्गोऽलिसङ्ख्यां वद ॥५५॥ ।शार्दूलविक्रीडितम्।

pañcāṃśo'likulātkadambamagamattryaṃśaḥ śilīndhraṃ tayoḥ
viśleṣastriguṇo mṛgākṣi kuṭajaṃ dolāyamāno'paraḥ |
kānte ketakamālatīparimalaprāptaikakālapriyā-
dūtāhūta itastato bhramati khe bhṛṅgo'lisaṅkhyāṃ vada ||55||

O doe-eyed [girl]! One-fifth from a group of bees went to the *Kadamba* [tree], one-third to the *Śilīndhra*, [and] three times their difference (*viśleṣa*) to the *Kuṭaja* [tree]. Another bee being enticed by the fragrances of the *Ketaka* and *Mālatī* flowers, like the one who has been approached by the messengers of two beloveds at the same time, is wavering and wandering here and there in the sky (*khe*). O my dear! State the [total] number of bees.

This verse presents a problem which requires solving a single variable linear equation using the aforementioned *iṣṭakarma*. Bhāskara elevates this mundane

mathematical problem through the use of imagery, and poetry imbibed with the *upamā-alaṅkāra*. After describing how a swarm of bees splits into various groups, each group seeking the nectar from the flowers of a different tree, he uses a simile to describe the confused state of a final single bee which is simultaneously attracted by the fragrances of two different flowers. He compares the state of this bee with that of a person who is summoned by two beloveds at the same time. The imagery, combined with the brilliant choice of *upamā*, as well the musical metre, results in one of the most beautiful verses of the *Līlāvatī*.

Bhāskara also addresses the student here as *mrgākṣi*, or doe-eyed girl, which is another instance of *upamā*. This is not an isolated instance where Bhāskara uses such beautiful words to address students. Elsewhere too we find several such instances. For example in verse 42, he challenges the student to solve the problem with the words *darbhīyagarbhāgrasutīkṣṇabuddhiḥ* (if your intellect is sharp as the inner tip of the *darbha* grass). In verse 17, he addresses the student as *bālakuraṅgalolanayane* or the one having darting eyes resembling those of a fawn.

Example 2

While dealing with the areas of various geometrical shapes, Bhāskara gives the following verse which succinctly presents three important formulae—the area of a circle, the surface area of a sphere, and the volume of a sphere.

वृत्तक्षेत्रे परिधिगुणितव्यासपादः फलं यत्
क्षुण्णं वेदैरुपरि परितः कन्दुकस्येव जालम् ।
गोलस्यैवं तदपि च फलं पृष्ठजं व्यासनिघ्नं
षड्भिर्भक्तं भवति नियतं गोलगर्भे घनाख्यम् ॥२०१॥ । मन्दाक्रान्ता ।

vṛttakṣetre paridhiguṇitavyāsapādaḥ phalaṃ yat
kṣuṇṇaṃ vedairupari paritaḥ kandukasyeva jālam |
golasyaivaṃ tadapi ca phalaṃ pṛṣṭhajaṃ vyāsanighnaṃ
ṣaḍbhirbhaktaṃ bhavati niyataṃ golagarbhe ghanākhyam ||201||

In a circular figure, the product of circumference and one-fourth of the diameter (*vyāsapāda*) gives the area (*phalam*), which multiplied by four gives the all round surface area of a sphere, which is similar to that of the net (*jāla*) all over (covering) the ball. Also, that surface area multiplied by the diameter (*vyāsanighnam*) and divided by six is known as the volume (*ghana*) that is confined inside the sphere.

The three formulae encoded in the above verse may be written as:

Figure 2: The surface of the sphere as visualised by Bhāskara.

$$\begin{aligned} \text{Area of a circle} \qquad & A_c = C \times \frac{D}{4} \\ \text{Surface area of a sphere} \qquad & S = A_c \times 4 = C \times D \\ \text{Volume of a sphere} \qquad & V = S \times \frac{D}{6}, \end{aligned}$$

where C and D are the circumference and diameter of the circle or sphere respectively.

But for presenting the formulae, the *Līlāvatī* doesn't elaborate on how the above results are derived. However, Bhāskara provides a brilliant demonstration of these results in his *Vāsanābhāṣya* commentary on the *Bhuvanakośa* chapter of his own *Siddhāntaśiromaṇi*. The proofs for the surface area and volume of a sphere in particular rely on the visualisation of a grid on the sphere, like a net on a ball, as shown in the accompanying figure. Therefore, the simile employed in this verse for visualising the grid gains in significance. This is because the surface of a sphere forms an example of a non-Euclidean surface whose area cannot be computed in a simple way. Once this surface is divided into small segments of familiar geometrical shapes (triangle and trapezia), then it becomes convenient to compute the area of these well known shapes and thereby get the desired surface area of the sphere. As the complete proofs—though ingenious and insightful—are beyond the scope of this paper, we shall conclude by stating that the use of *upamā* in this verse is a crucial indicator of Bhāskara's thought process and serves an important purpose in his proof.

5.2.2 *Ullāsa*

The *ullāsa-alaṅkāra* is used to show the greatness of something by showing deficiency elsewhere. *Candrāloka* defines it with a beautiful example as follows [CaLo1950, p. 174]:

उल्लासोऽन्यमहिम्ना चेद्दोषो ह्यन्यत्र वर्ण्यते ।
तदभाग्यं धनस्यैव यन्नाश्रयति सज्जनम् ॥

ullāso'nyamahimnā ceddoṣo hyanyatra varṇyate |
tadabhāgyaṃ dhanasyeva yannāśrayati sajjanam ||

That is *ullāsa*, where by the virtue of greatness of something, a fault is described in something else. [Ex.] It is only the bad luck of that wealth that it does not take refuge in a noble person!

The example accompanying the above definition seeks to illustrate the greatness of noble persons (who are presumably of modest means) by presenting their lack of means as the misfortune of the wealth! Use of this *alaṅkāra* is observed in the following verse of the *Līlāvatī*:

राश्योर्ययोः कृतिवियोगयुती निरेके
मूलप्रदे प्रवद तौ मम मित्र यत्र ।
क्लिश्यन्ति बीजगणिते पटवोऽपि मूढाः
षोढोक्तगूढगणितं[15] परिभावयन्तः ॥६२॥ । वसन्ततिलका ।

rāśyoryayoḥ kṛtiviyogayutī nireke
mūlaprade pravada tau mama mitra yatra |
kliśyanti bījagaṇite paṭavo'pi mūḍhāḥ
ṣoḍhoktagūḍhagaṇitaṃ paribhāvayantaḥ ||62||

O friend! State those two [numbers], the sum and difference of whose squares reduced by one result in square numbers, wherein even experts in algebra who keep dwelling upon (*paribhāvayantaḥ*) the subtle (*gūḍha*) mathematical techniques stated in six ways, being thoroughly confused, face difficulty [in solving this problem].

In the first part of this verse, Bhāskara demands the solution to the two following algebraic equations:

$$a^2 + b^2 - 1 = S_1$$
$$b^2 - a^2 - 1 = S_2,$$

[15] The reading in [Līlā1937, p. 58] is षोढोक्तबीजगणितं. However, the same text also notes the reading given above in a footnote. The *Kriyākramakarī* [Līlā1975, p. 136] too prefers the given reading, which we think is more suitable as it is alliteratively appropriate, and fits in the sequence मूढाः–षोढ–गूढ. It may also be noted that the same terms are also used in verse 64 of the *Līlāvatī* as well.

where S_1 and S_2 are squares, and $b > a$.

It is in the second part of the verse that we encounter the *ullāsa-alaṅkāra*. The 'six subtle mathematical techniques' mentioned here are found in the first chapter of Bhāskara's own *Bījagaṇita*, called *avyaktaṣaḍvidha* (six methods of dealing with unknowns). This chapter describes six mathematical operations including addition, subtraction, multiplication, division, squares, and square roots of unknowns.[16] Bhāskara then states that even those skilled in these algebraic operations would find difficulty in solving the above problem. Thereby, Bhāskara illustrates the great difficulty of algebra by portraying even the experts as inept at it. This therefore is an example of *ullāsa*.

5.2.3 *Kāvyaliṅga*

The *kāvyaliṅga-alaṅkāra* enhances the beauty of a verse by implicitly making use of some inside information which the reader is reasonably supposed to possess, the knowledge of which assists in obtaining a deeper understanding of the verse. The *Candrāloka* defines this *alaṅkāra* with an example as follows [CaLo1950, p. 124] :

> स्यात् काव्यलिङ्गं वागर्थो नूतनार्थसमर्पकः ।
> जितोऽसि मन्दकन्दर्प मच्चित्तेऽस्ति त्रिलोचनः ॥
>
> *syāt kāvyaliṅgaṃ vāgartho nūtanārthasamarpakaḥ |*
> *jito'si mandakandarpa maccitte'sti trilocanaḥ ||*
>
> When the express sense of the words (*vāgartha*) [employed in the verse] presents a [cue to a] new meaning then it is [an instance of] *kāvyaliṅga*. [Ex.] O dull-witted Cupid! You have been won over by me. In my heart resides the three-eyed Lord (Śiva).

Two facts are presented by the express sense of the words used in the above verse: (i) victory over cupid, and (ii) the Lord Śiva residing in one's heart. Here, the knowledge of the *puraṇic* story of Śiva's defeat of Kāma (cupid) enhances the beauty of the verse as the reader is then aware as to why it is possible to win over cupid when Śiva resides in the heart.

A similar fore-knowledge of certain facts enhances the beauty of the following invocatory verse from the *Līlāvatī*:

[16] One may wonder why these techniques have been described as 'subtle'. In addition to allowance for poetic liberties, it should be noted that for much of human history, algebra has been an esoteric science studied only by a few. Mathematical operations with variables or unknowns were neither well understood nor easily solved. Therefore, in this particular context, these methods were certainly cryptic and subtle.

लीलागललुलल्लोलकालव्यालविलासिने ।
गणेशाय नमो नीलकमलामलकान्तये ॥९॥ । अनुष्टुभ् ।

līlāgalalulallolakālavyālavilāsine
gaṇeśāya namo nīlakamalāmalakāntaye ||9||

Salutations to Gaṇeśa, who is resplendent like the spotless (*amala*) blue lotus, and who is playing with a black serpent which is gracefully swaying (*lulat*), coiling and uncoiling (*lolat*) around the neck.

Here, a blue-hued Gaṇeśa is described as playing with a black snake hung around the neck. However, in the Indian tradition, Gaṇeśa is not usually depicted with a snake around the neck. Moreover, he is usually described as fair-complexioned (*śaśivarṇa*), rather than blue-hued. Here, the additional knowledge of the fact that Gaṇeśa is the son of *Śiva*, who is adorned with a snake around his neck, leads to a possible interpretation that Gaṇeśa is playing with the snake coiled around his father's neck, the reflection of whose colour on his fair skin is causing him to take on a blue hue.

Note that the description of Gaṇeśa as 'resplendent like the spotless blue lotus' (*nīlakamalāmalakāntaye*) is another example of the *upamā-alaṅkāra*.

5.2.4 *Tadguṇa*

The *tadguṇa-alaṅkāra* occurs when one entity is described to have taken the property of another entity, by virtue of some association with the latter. *Candrāloka* defines the *tadguṇa* with an example as follows [CaLo1950, p. 175]:

तद्गुणः स्वगुणत्यागात् अन्यतस्स्वगुणोदयः ।
पद्मरागारुणं नासामौक्तिकं तेऽधरश्रितम् ॥

tadguṇaḥ svaguṇatyāgāt anyatassvaguṇodayaḥ |
padmarāgāruṇaṃ nāsāmauktikaṃ te'dharaśritam ||

Tadguṇa is said to occur, where [an entity] is described to acquire other's quality as its own, having shed own quality. [Ex.] Because of being in the proximity of your lower lip, the pearl in your nose looks as red as a ruby.

In the above example, the pearl in the nose of a lady loses its pearly lustre, and appears like a ruby, having taken on the redness of the lip. Similarly, in the above invocatory verse of the *Līlāvatī*, Gaṇeśa takes on a blue-hue due to the reflection of the colour of the snake he is playing with.

5.2.5 *Svabhāvokti*

Svabhāvokti is used to describe the characteristics of any entity. *Candrāloka* defines it as follows [CaLo1950, p. 181]:

स्वभावोक्तिः स्वभावस्य जात्यादिषु च वर्णनम् ।

svabhāvoktiḥ svabhāvasya jātyādiṣu ca varṇanam |

Svabhāvokti is [the *alaṅkāra*], when we find an enticing description of the generic characteristics of an entity etc.

The following verse of the *Līlāvatī* from the chapter on quadratic equations forms a beautiful example of *svabhāvokti*:

अलिकुलदलमूलं मालतीं यातमष्टौ
निखिलनवमभागाश्चालिनी भृङ्गमेकम् ।
निशि परिमललुब्धं पद्ममध्ये निरुद्धं
प्रतिरणति रणन्तं ब्रूहि कान्तेऽलिसंख्याम् ॥७१॥ । मालिनी ।

alikuladalamūlaṃ mālatīṃ yātamaṣṭau
nikhilanavamabhāgāścālinī bhṛṅgamekam |
niśi parimalalubdhaṃ padmamadhye niruddhaṃ
pratiraṇati raṇantaṃ brūhi kānte'lisaṃkhyām ||71||

The square root of half of a swarm of bees and [also] eight-ninths of the total went to the *mālatī* [tree]. A female bee is buzzing in response to the buzzing of a bee which attracted by the fragrance, got trapped in a lotus at night. O charming lady! State the [total] number of bees.

In this verse, we need to determine the total number of bees in a swarm, all of which go to the *mālatī* flower (in different groups), except for two bees. Of these two, one is trapped in a lotus, and the other bee is buzzing in response to the cries of the trapped bee. Bees are known to be attracted to the fragrance of flowers, and are also reputed for their buzzing noise. Since the description of the natural characteristics of a bee—*parimalalubdham* ('entranced by the fragrance'), and *raṇantam* (buzzing)—has been done in an enchanting manner, this verse is a good example of the use of *svabhāvokti.*

The problem given in the verse can be solved as follows. Let the total number of bees be $2x^2$. Then according to the problem, x bees as well eight-ninths of $2x^2$ went to the tree with *Mālatī* flowers. We also have two more bees—the female bee, and the bee trapped in the lotus. Therefore,

$$2x^2 = x + \frac{8}{9} \times 2x^2 + 2$$

$$\text{or,} \qquad x^2 - \frac{9}{2}x = 9.$$

Solving the quadratic equation using a technique described by Bhāskara in the *Līlāvatī*, we have

$$x^2 = \left[\sqrt{9 + \left(\frac{1}{2} \times \frac{9}{2}\right)^2} + \frac{1}{2} \times \frac{9}{2}\right]^2 = 36.$$

Therefore, the total number of bees is $2x^2 = 72$.

5.2.6 *Śleṣa*

The *śleṣa* occurs when it is possible to interpret a verse, or a part of it, in more than one way. *Candrāloka* states [CaLo1950, p. 142]:

नानार्थसंश्रयः श्लेषः ।

nānārthasaṃśrayaḥ śleṣaḥ |

Śleṣa occurs where the words have various meanings.

Bhāskara outdoes himself in the following final verse from *Līlāvatī* where he displays his poetic genius by employing a variant of *śleṣa* called *khaṇḍaśleṣa*, such that the verse can be interpreted in two ways:

येषां सुजातिगुणवर्गविभूषिताङ्गी
शुद्धाखिलव्यवहृतिः खलु कण्ठसक्ता ।
लीलावतीह सरसोक्तिमुदाहरन्ती
तेषां सदैव सुखसंपदुपैति वृद्धिम् ॥२७२॥ । वसन्ततिलका ।

yeṣāṃ sujātiguṇavargavibhūṣitāṅgī
śuddhākhilavyavahṛtiḥ khalu kaṇṭhasaktā |
līlāvatīha sarasoktimudāharantī
teṣāṃ sadaiva sukhasaṃpadupaiti vṛddhim ||272||

Here [in this world], those for whom the *Līlāvatī*—whose sections (*aṅga*) are adorned with procedures for reductions of fractions (*jāti*), rules for multiplication (*guṇa*), squaring (*varga*) [etc.], which has descriptions that are faultless, [and] which presents elegant and enchanting examples—is memorised, for them the wealth of happiness will indeed always increase.

Firstly, the given verse can be interpreted as shown above, where Bhāskara highlights the mathematical aspect of *Līlāvatī* and states that whoever memorises this text, he or she would thoroughly and progressively enjoy the beauty of its contents as well as its presentation. An alternative interpretation of the verse which conflates *Līlāvatī* with a beautiful woman, whose company brings joy is shown below:

> Here [in this world], a beautiful woman who—is high-born and adorned with many virtues, having pure and blemishless conduct, [and] who utters enticing words—is in the embrace of whosoever, their wealth of happiness will indeed always increase.

The alternate interpretations of the phrases employed in the verse which make this different reading possible are shown in Table 4. This verse only further demonstrates the poetic genius of Bhāskara, and the employment of *śleṣa* in the final verse is definitely the supreme expression of Bhāskara's skill in poetry.

Table 4: Alternate interpretation of verse 272.

Phrase	Main reading	Alternate reading
कण्ठसक्ता	memorised	whose neck is embraced by
लीलावती	the text *Līlāvatī*	the beautiful woman
सुजाति	reduction of fractions	qualities of being high born
गुणवर्ग	rules of multiplication and squaring	multitude of virtues
शुद्धाखिलव्यवहृतिः	having all faultless descriptions	having pure and blemishless conduct
सरसोक्तिमुदाहरन्ती	presenting enchanting examples	utters enticing words

6 The mathematics of *Līlāvatī*

6.1 An overview of its content

Bhāskara in the very first invocatory verse, indicates that the *Līlāvatī* is intended to be a good introductory text book on mathematics (*sadgaṇitasya pāṭīṃ vacmi*). Typically a text book on mathematics in the Indian tradition commences with *paribhāṣā* or terminology, and then introduces the *parikarmas* or basic arithmetic operations—with whole numbers as well as fractions. The discussion on *parikarma* includes a variety of topics such as finding square, square root, cube, cube root, rule of three and its inverse (*vyastatrairāśika*), rule of five, and so on.

Then the text proceeds to deliniate procedures to deal with progressions (arithmetic and geometric), and a variety of other problems known as *miśravyavahāra*, that has to do primarily with arithmetic calculations. This includes problems related to calculation of interest, sharing of profits among investors, mixing of gold etc. The wide array of problems included under this section depends upon the predilections of the author and the demand of the day. After discussing problems that have to do with arithmetic, ranging in difficulty from easy to hard, the author proceeds to deal with *kṣetravyavahāra* or geometry. Bhāskara has devoted almost one fourth of his text (verses 135–213) to deal with geometry at great length. He has also discussed problems related to calculation of area and volume, first order indeterminate equations, and permutations. Table 5 enumerates the wide range of topics discussed in the *Līlāvatī*. Thus, the *Līlāvatī* can be considered as a primer in mathematics that is complete by itself. A systematic study of this text should suffice to bring one up to the level of a modern day high school student.

Table 5: The different topics discussed in the *Līlāvatī*.

No.	**Sanskrit**	**English**	**Verses**
1	परिभाषा	Terminology	1–8
2	परिकर्माष्टकम्	Eight types of operations	9–29
3	भिन्नपरिकर्माष्टकम्	Eight types of operations with fractions	30–44
4	शून्यपरिकर्माष्टकम्	Eight types of operations with zero	45–47
5	प्रकीर्णकानि	Assorted rules	48–89
6	मिश्रव्यवहारः	Dealing with mixed quantities	90–116
7	श्रेढीव्यवहारः	Dealing with series	117–134
8	क्षेत्रव्यवहारः	Dealing with planar figures	135–213
9	खातव्यवहारः	Dealing with excavations	214–219
10	चितिव्यवहारः	Dealing with altars	220–222
11	क्रकचव्यवहारः	Dealing with sawing	223–226
12	राशिव्यवहारः	Dealing with heaps	227–231
13	छायाव्यवहारः	Dealing with shadows	232–241
14	कुट्टकव्यवहारः	Dealing with the pulveriser	242–260
15	अङ्कपाशः	Permutations	261–272

6.2 Bhāskara's style of presentation

Covering a wide array of topics in a systematic manner—as indicated in the previous section—alone would not have led to the *Līlāvatī* being widely adopted as the standard text book of mathematics in the traditional educational system in India, for almost 700–800 years, before British intervention.[17] The *Līlāvatī* is also an intensely practical text, steadfastly focusing on real life problems, instead of talking abstract mathematics in a vacuum. To this end, after dealing with the very basic operations of mathematics in the first half, Bhāskara categorises topics according to application (like dealing with altars, excavations, sawing of wood etc.) in the second half. As the readers will observe, a lot of this mathematics deals with the problems concerned with the life of ordinary people, and is not restricted to merely concerns of the elite who presumably may only be interested in the mathematics of astronomy and such topics. This highlights the ill-informed nature of the charges of elitism levelled by some against the Indian educational system of yore.

It cannot be denied that in a few instances, the text is somewhat dense and difficult to understand. In many such cases, the nature of the mathematical operation being described itself is quite complicated. To ameliorate such situations, where felt necessary, Bhāskara has invariably elucidated the verses in prose in his auto-commentary *Vāsanābhāṣya*, and also demonstrated the steps involved in the operation by working out an example. This can be illustrated with an example from the chapter on quadratic equations (referred to as the *guṇakarma*).

Quadratic equations are univariate second order equations, usually represented in the form

$$ax^2 + bx + c = 0 \tag{3}$$

in modern mathematics. These equations have been studied in India from antiquity, with Brahmagupta prominently addressing them in his *Brāhmasphuṭasiddhānta*. Bhāskara too addresses problems of this type, but in contrast to the given equation, he frames his equation in the form

$$x^2 \mp bx = c, \tag{4}$$

where x^2 is referred to as *rāśi*, b is called *guṇa* or *mūlaguṇa*, and c is *dṛṣṭa* or *dṛśya*. The coefficient of x^2, a, is taken to be one. Bhāskara arrives at the solution to these kinds of problems by modifying the equation by adding the square of half the *guṇa* to both sides, as shown below. Therefore, he refers to

[17] In fact, even today, we find the text being studied in a few traditional schools across India.

this operation as *guṇakarma*. With this terminology in mind, let's look at the way Bhāskara presents the solution to a quadratic equation.

> गुणघ्नमूलोनयुतस्य राशेर्दृष्टस्य युक्तस्य गुणार्धकृत्या ।
> मूलं गुणार्धेन युतं विहीनं वर्गीकृतं प्रष्टुरभीष्टराशिः ॥६५॥[18]
>
> *guṇaghnamūlonayutasya rāśerdṛṣṭasya yuktasya guṇārdhakṛtyā |*
> *mūlaṃ guṇārdhena yutaṃ vihīnaṃ vargīkṛtaṃ praṣṭurabhīṣṭarāśiḥ* ||65||
>
> That [desired intangible] quantity (*rāśi*) which has become tangible by subtracting or adding the product of a multiplier (*guṇa*) and [its own] square-root, is added to the square of half the multiplier and the square root is taken. Half the multiplier, when added to or subtracted from this, when squared, gives the [intangible] quantity sought by the questioner.

In contrast to modern convention, Bhāskara considers x^2 to be the unknown quantity (*rāśi*), instead of x. In the above verse, Bhāskara gives the following solution to (4):

$$x^2 = \left[\sqrt{c + \left(\frac{b}{2}\right)^2} \pm \frac{b}{2}\right]^2. \qquad (5)$$

It is not so easy to comprehend the above equation from the verse given, as it has been presented in a terse form. The following passage in the auto-commentary of Bhāskara greatly facilitates the reader in comprehending the above verse.

> यो राशिः स्वमूलेन केनचिद्गुणितेनोनो दृष्टः तस्य मूलगुणार्धकृत्या युक्तस्य यत्पदं तद्गुणार्धेन युक्तं कार्यम्। यदि गुणघ्नमूलयुतो दृष्टः तर्हि हीनं कार्यम्। तस्य वर्गो राशिः स्यात्।
>
> *yo rāśiḥ svamūlena kenacidguṇitenono dṛṣṭaḥ tasya mūlaguṇārdhakṛtyā yuktasya yatpadaṃ tadguṇārdhena yuktaṃ kāryam yadi guṇaghnamūlayuto dṛṣṭaḥ tarhi hīnaṃ kāryam tasya vargo rāśiḥ syāt* |
>
> The [intangible] quantity which has become tangible by diminishing it by the product of any multiplier (*guṇa*) and [its own] square-root, is added to the square of half the multiplier, and the square root is taken. The square-root of that should be added by half the multiplier. If [the intangible *rāśi* became] tangible by adding the product of the multiplier and the root (*mūla*), then [that square-root] should be diminished [by half the multiplier]. The square of that would be the [desired] quantity.

One could lament that the prose quoted above is also quite tightly written, and doesn't explain the problem or the solution in an elaborate manner. But hardly anything can be done about it, as Bhāskara's style of writing is quite succinct —be it prose or poetry—upholding the age-old principle:

[18] The verse in prose form with the necessary supply of words: यः [अव्यक्तः अभीष्टो] राशिः गुणघ्नमूलोनयुतः [सन्] दृष्टः जातः, तस्य [दृष्टस्य राशेः] गुणार्धकृत्या युक्तस्य मूलं [कार्यम्। तत्] गुणार्धेन युतं विहीनं [वा कार्यम्। तद्] वर्गीकृतं प्रष्टुरभीष्टराशिः [भवेत्]।

अर्धमात्रालाघवेन पुत्रोत्सवं मन्यन्ते ।

ardhamātrālāghavena putrotsavaṃ manyante |

If they could save even half a syllable, they celebrate it like the birth of a son.

Nevertheless, one cannot deny the fact that the auto-commentary *Vāsanābhāṣya* quoted above elucidates the contents of the verse and helps the reader in comprehending the verse more easily.

The expression for x^2 given above can be derived as follows. Adding $\left(\frac{b}{2}\right)^2$ to both sides of (4) as stated by Bhāskara in the verse, we have

$$\begin{aligned} x^2 \mp bx + \left(\frac{b}{2}\right)^2 &= c + \left(\frac{b}{2}\right)^2 \\ \left(x \mp \frac{b}{2}\right)^2 &= c + \left(\frac{b}{2}\right)^2 \\ x \mp \frac{b}{2} &= \sqrt{c + \left(\frac{b}{2}\right)^2} \\ x &= \sqrt{c + \left(\frac{b}{2}\right)^2} \pm \frac{b}{2} \\ \therefore x^2 &= \left[\sqrt{c + \left(\frac{b}{2}\right)^2} \pm \frac{b}{2}\right]^2 . \end{aligned}$$

In modern mathematical texts, the roots of a quadratic equation of the form $ax^2 + bx + c = 0$ are usually represented as

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

Bhāskara's expression (5) can also be simplified and re-written as follows:

$$x = \frac{\sqrt{b^2 + 4c} \pm b}{2}.$$

While the two expressions seem different, by adjusting for the difference in signs of b and c in (3) and (4), and putting $a = 1$, we can show that they are exactly the same. However, it is important to note that Bhāskara has presented only one root to the quadratic equation. The $\pm$ sign seen in (5) is only to adjust for the sign of b in (4), and not to suggest two roots to the equation. Therefore, the sign of b in the solution is the reverse of that in the problem. A beautiful example illustrating the application of the solution to

the quadratic equation discussed above, has already been discussed in the fifth example in Section 5.1.

6.3 *The uniquenes of the Līlāvatī from a mathematical standpoint*

In composing the *Līlāvatī*, Bhāskara would have naturally been influenced by the works of earlier astronomer-mathematicians. Indeed, we find a number of rules in the *Līlāvatī* very similar to those proposed by Brahmagupta in the *Brāhmasphuṭasiddhānta*, and Śrīdhara in the *Triśatikā*. He also borrows examples from prior works, and indeed states so explicitly. Hence, strictly speaking, purely from the view point of its mathematical content, the *Līlāvatī* is more of an incremental improvement rather than a revolutionary change over earlier works.

However, we find that in handling many topics, Bhāskara clearly improves upon the rules of his predecessors (where possible), either by making them more efficient, distinctly delineating the scope of application of the rule, or by presenting them more succinctly, or by combining them suitably. Bhāskara also introduces a new topic called *aṅkapāśa* or permutations (distinctly from combinations), which to our knowledge has not been dealt with by any other text in the Indian tradition till his time. In this chapter, Bhāskara deals with various rules of permutations, largely pertaining to their application in estimating the count and sum of numbers which can be formed from a given set of digits. As this topic is quite novel and interesting, we devote the following section to give the readers a flavour of this topic.[19]

6.3.1 Permutations

Permutations are a measure of the number of ordered ways in which a subset of objects can be selected from a given collection. Bhāskara deals with various rules of permutations, largely pertaining to their application in estimating the count and sum of numbers which can be formed from a given set of digits. He commences the chapter called *aṅkapāśa* with the basic problem of determining the number of possible permutations in selecting all the objects from a given collection, i.e., determining the number of ways to order n objects.

[19] A beautiful example from this chapter, dealing with the iconography of Viṣṇu and Śiva *mūrtis*, has already been discussed in the third example in Section 3.2.

The verse below furnishes two basic rules of permutation. First, given n distinct digits (except zero), the verse describes how to determine the count of numbers which can be formed using all of the digits. Secondly, the verse describes how to determine the sum of all the numbers so formed.

> स्थानान्तमेकादिचयाङ्कघातः संख्याविभेदा नियतैस्स्युरङ्कैः ।
> भक्तोङ्कमित्याङ्कसमासनिघ्नः स्थानेषु युक्तो मितिसंयुतिस्स्यात् ॥२६१॥
>
> *sthānāntamekādicayāṅkaghātaḥ saṃkhyāvibhedā niyataissyuraṅkaiḥ |*
> *bhaktoṅkamityāṅkasamāsanighnaḥ sthāneṣu yukto mitisaṃyutissyāt* ||261||
>
> The product of the numbers starting with and increasing by one, unto the [given number of] places, would be the varieties of numbers (*saṅkhyāvibheda*) [that can be formed] with [all of] the given digits. That [*saṅkhyāvibheda*] divided by the number of digits, multiplied by the sum of the digits, and [the result] added in place [according to place value], would be the sum of the [various] numbers formed.

Given n 'distinct' digits, it is said that the count of numbers which can be formed using all the digits is the product of all the numbers from 1 to n. Representing the permutation by the notation ${}^nP_n$, and using the factorial notation (!) to represent the product of the first n integers, the rule given in the verse can be represented mathematically as follows:

$$ {}^nP_n = n! = 1 \times 2 \times \cdots \times n. \tag{6}$$

The latter part of the verse describes how to calculate the sum of these $n!$ numbers formed above. The calculation of this sum involves two steps: (i) finding an intermediary number s', and (ii) adding s' unto itself n times, by sliding the number to successive higher place value each time starting from the units place.

Let s_n (referred to as *aṅkasamāsa* in the verse) be the sum of the n digits of the given number. Representing the n digits by $d_1, d_2, \ldots, d_n$, we have

$$s_n = d_1 + d_2 + \cdots + d_n.$$

Then, as the first step, we calculate the intermediary value s' using the rule given by Bhāskara in the third quarter of the verse as follows:

$$s' = \frac{n! \times s_n}{n}. \tag{7}$$

In the second step, we need to add s' to itself n times, adjusting each instance to a higher place value. That is, we need to successively place s' in the units place, tens place, and so on unto the n^{th} place, and add the resulting

numbers.[20] The result of this operation gives us the sum S of all the numbers generated through the permutation.

The rationale behind this method can be understood as follows. The $n!$ numbers generated through permutation have n digits each. Comparing the units place of all these numbers, we find that each of the digits occur $(n-1)!$ times.[21] Similarly, each digit occurs $(n-1)!$ times in each position. Therefore, upon adding all the numbers in any given column, we obtain the total:

$$(n-1)! \times (d_1 + d_2 + \cdots + d_n) = (n-1)! \times s_n.$$

This is the same intermediary value given in (7). Now, knowing the sum of the columns, we can determine the sum of the $n!$ numbers by adjusting these sums for place value and adding them. Thus, we arrive at the sum of all the numbers of the permutation.

The above procedure can be illustrated by taking $n = 3$, and d_1, d_2, d_3 as the three given digits. Upon permutation, we get the six numbers $d_1d_2d_3$, $d_1d_3d_2$, $d_2d_1d_3$, $d_2d_3d_1$, $d_3d_1d_2$, and $d_3d_2d_1$. Adding all these numbers, we obtain the sum $2 \times (d_1 + d_2 + d_3) = 2 \times s_n$ in the units, tens, as well as the hundreds place.[22] Adjusting for place value and adding, the summation of the six numbers equals

$$S = 2s_n \times (1 + 10 + 100) = 2s_n \times 111.$$

Extending the above argument, the general formula for the sum of the numbers obtained from the permutation can be given as:[23]

$$\begin{aligned} S &= (n-1)! \times s_n \times (10^0 + 10^1 + \cdots + 10^{n-1}) \\ &= (n-1)! \times s_n \times \frac{10^n - 1}{9}. \end{aligned}$$

Nārāyaṇa gives the following equivalent expression in *Kriyākramakarī* for calculating the sum of the permutations:[24]

$$S = \frac{n! \times s_n}{n} \times (111 \ldots\ n \text{ ones.}) \tag{8}$$

Having given the rule for determining the number of possible permutations of a given number, as well as finding the sum of all the numbers thus gener-

[20] Readers may refer to the example in the next verse for an illustration of this method.

[21] Given $n!$ numbers, each of the n digits occurs $\frac{n!}{n} = (n-1)!$ times in any given position.

[22] Note that here we have $\frac{n!}{n} = \frac{3!}{3} = 2$.

[23] Here we use the general relation $k^1 + k^2 + \cdots + k^n = \frac{k^n - 1}{k - 1}$.

[24] [Līlā1975, p. 459].

ated, Bhāskara immediately presents the following problem as an illustrative example.

> द्विकाष्टकाभ्यां त्रिनवाष्टकैर्वा निरन्तरं द्व्यादिनवावसानैः ।
> संख्याविभेदाः कति संभवन्ति तत्संख्यकैक्यानि पृथग्वदाशु ॥२६२॥ । उपजातिः ।
>
> *dvikāṣṭakābhyāṃ trinavāṣṭakairvā nirantaraṃ dvyādinavāvasānaiḥ |*
> *saṃkhyāvibhedāḥ kati saṃbhavanti tatsaṃkhyakaikyāni pṛthagvadāśu* ||262||
>
> How many varieties of numbers are possible with [the digits] two and eight; three, eight and nine; and [the digits] starting with two and ending with nine without interval (2 to 9)? Separately, quickly tell the sums of those numbers [in each of the above cases].

In this problem, we need to determine the number of permutations possible with the each of the following three sets of digits: [2 8], [3 8 9], and [2 3 4 5 6 7 8 9]. Additionally, we also need to determine the sums of the numbers thus formed. Here, the number of digits (n) in each of the above sets is 2, 3, and 8 respectively. Applying (6), the number of permutations possible for each set is as follows:

$$
\begin{aligned}
{}^2P_2 &= 1 \times 2 = 2 \\
{}^3P_3 &= 1 \times 2 \times 3 = 6 \\
{}^8P_8 &= 1 \times 2 \times 3 \times 4 \times 5 \times 6 \times 7 \times 8 = 40320.
\end{aligned}
$$

That is, 2, 6, and 40320 numbers can be formed by the permutation of the digits in each of the sets respectively.

To solve for the sum of the numbers obtained from the permutation, we first determine the sum of the digits in each set. Therefore, we have $s_2 = 2+8 = 10$, $s_3 = 3+8+9 = 20$, and similarly, $s_8 = 44$. We next determine the intermediary value s' in each case using (7):

$$
s'_2 = \frac{2 \times 10}{2} = 10 \qquad s'_3 = \frac{6 \times 20}{3} = 40 \qquad s'_8 = \frac{40320 \times 44}{8} = 221760.
$$

We then obtain the sum of the numbers generated from the permutation by placing s' successively in the units place (U), tens place (T), and so on, and adding the resulting numbers. This process is shown in Figure 3. As shown, we obtain the results $S_2 = 110$, and $S_3 = 4440$.

In the case of the numbers generated from the set of eight digits, using (8) we have:

$$
S_8 = \frac{8! \times 44}{8} \times 11111111 = 2463999975360.
$$

		T	U
	1	0	
+		1	0
	1	1	0

(a) Sum of the numbers generated by permutation of [2 8]

		H	T	U
	4	0		
+		4	0	
+			4	0
	4	4	4	0

(b) Sum of the numbers generated by permutation of [3 8 9]

Figure 3: Sum of the numbers generated by permutation using Bhāskara's method.

6.3.2 Operations involving division by zero

Zero is a unique number with special properties and behaves differently from any other number when participating in mathematical operations like multiplication, division etc. Therefore, mathematical operations involving zero have been dealt with separately by various Indian mathematicians, and Bhāskara too follows this pattern. While discussing the various mathematical operations with zero (*kham* or *śūnya*), Bhāskara in his *Lilāvatī* states that if a finite quantity has zero as its multiplier, and also as its divisor, then the resulting number would be the original quantity itself.

योगे खं क्षेपसमं वर्गादौ खं खभाजितो राशिः ।
खहरस्स्यात् खगुणः खं खगुणश्चिन्त्यश्च शेषविधौ ॥४५॥
शून्ये गुणके जाते खं हारश्चेत् पुनस्तदा राशिः ।
अविकृत एव ज्ञेयस्तथैव खेनोनितश्च युतः ॥४६॥ । आर्या ।

yoge khaṃ kṣepasamaṃ vargādau khaṃ khabhājito rāśiḥ |
khaharassyāt khaguṇaḥ khaṃ khaguṇaścintyaśca śeṣavidhau ||45||
śūnye guṇake jāte khaṃ hāraścet punastadā rāśiḥ |
avikṛta eva jñeyastathaiva khenonitaśca yutaḥ ||46||

In addition, zero becomes equal to the additive. In squaring etc., [zero gives] zero. Any quantity divided by zero would become *khahara* [zero-denominator]; multiplied by zero would become zero, and should [instead] be considered *khaguṇa* [zero-multiplied] in case of any remaining operation. When zero happens to be the multiplier, and again zero is the divisor, then the quantity should be known to be unchanged only. Similarly, a quantity deducted and added by zero [should be known to be unchanged.]

This operation described by Bhāskara above has been criticised by some modern commentators of the *Līlāvatī*. However, recently, a few scholars have argued for setting aside pre-conceived notions about 'right' and 'wrong' math-

ematics, and to study Bhāskara on his own merits, and in the context of his time. For instance, [Oko2017] not only justifies the result given by Bhāskara, but extols him as follows:[25]

> It may well make us ashamed that a 1152 AD mathematician should have attained to such a degree of knowledge as to see that $\frac{a}{0}$ is a number and not undefined.

Recently, by invoking the concept of idempotents that is employed in abstract algebra, [Sat2015] has proposed an explanation for Bhāskara's treatment of the conception of infinity (*khahara*) or division by zero. He concludes:

> We see that Bhāskarācārya certainly had a novel calculation scheme introduced in his exercises and might have intended further developments. However, he seems to have worked more extensively on astronomy and perhaps did not return to these ideas again. The idea about infinity and especially the idea of using this extended number system does seem to point to possible new algebraic concepts. Whether these ideas can create new useful Mathematical Systems remains an open question.

Bhāskara is certainly on firmer footing when he terms the result obtained by dividing a quantity by zero as *khahara*—literally 'zero-divisor'.[26] In the *Bījagaṇita*, he describes the immutable nature of *khahara*, and compares it to infinity using a philosophical concept:

अस्मिन्विकारः खहरे न राशावपि प्रविष्टेष्वपि निःसृतेषु ।
बहुष्वपि स्याल्लयसृष्टिकालेऽनन्तेऽच्युते भूतगणेषु यद्वत् ॥६॥

asminvikāraḥ khahare na rāśāvapi praviṣṭeṣvapi niḥsṛteṣu |
bahuṣvapi syāllayasṛṣṭikāle'nante'cyute bhūtagaṇeṣu yadvat ||6||

In this quantity *khahara*, there is no alteration, even when many [quantities] enter or exit, just as in the case of the infinite *Acyuta* at the time of destruction and creation, when many beings dissolve into and emanate from Him.

From this verse, it is apparent that Bhāskara possessed a keen understanding of division by zero, as well as the concept of infinity, much in line with

[25] Furthermore, the paper leads to novel insights with application in the foundation of differential calculus.

[26] This term has earlier been employed by Śrīpati in his *Siddhāntaśekhara* [SiŚe1947, p. 92] to describe a quantity divided by zero. However, confusingly, in his *Gaṇitatilaka* he describes the result of this operation as zero:

व्योम्ना भक्ते भवति गगनं व्योम्नि भक्ते च शून्यम् ।

vyomnā bhakte bhavati gaganaṃ vyomni bhakte ca śūnyam |

[A quantity] when divided by zero, becomes zero. When zero is divided, it becomes zero.

Brahmagupta in the *Brahmasputasiddhānta* [BSS1902, p. 310] denotes this quantity as *taccheda*.

the current understanding of the concept. Indeed, Bhāskara is probably the first Indian mathematician to equate *khahara* to infinity. Commenting upon *khahara*, Munīśvara argues that as the divisor decreases, the result increases, and therefore as the *khahara* has the least possible divisor, the division will result in a 'large quantity' (*phalaparamatva*). He notes that stating a value will diminish the largeness of this quantity since, there will always be a value higher than it, and concludes that the term *khahara* is therefore used to denote the result of division by zero.[27]

6.3.3 Pithy remarks

Yet another aspect that surely singles out Bhāskara's works has to do with the *pertinent* and *pithy* remarks that he makes in certain places in order to convey the importance of certain mathematical principles or processes. We demonstrate by considering a specific example.

In India and other civilisations around the world, measurement of shadows played a crucial role in fixing time as well as determination of various geographical and astronomical parameters like latitude, east-west line etc. Given the importance of this topic, Bhāskara in his *Līlāvatī* too discusses a few problems involving the measurement of shadows, which though appearing to be elementary, have critical applications in astronomy.

Shadows were usually measured by vertically erecting a *śaṅku* (stick or gnomon) in an open area with a flat surface to ensure accuracy. Verse 236 of the *Līlāvatī* describes a method to determine the height of the lamp, given the length of the shadow, as well as the distance between the lamp and the *śaṅku*. This verse below describes a method to determine (i) the distance from

[27] Munīśvara's original comment [Ms-Līla, f. 81v] is given below:

> हरापचये फलोपचयात् परमहरापचयशून्यमितहरस्य फलपरमत्वम्। इयत्तायाः परमत्वव्याघातः ततोऽप्याधिक्यसंभवात्। अत एवानन्तफलज्ञापनार्थं खहरत्वमुक्तम्।
> *harāpacaye phalōpacayāt paramaharāpacayaśūnyamitaharasya phalaparamatvam iyattāyāḥ paramatvavyāghātaḥ tatō'pyādhikyasaṃbhavāt ata evānantaphalajñāpanārthaṃ khaharatvamuktam* |
> Since when the divisor decreases, the result increases, the result would be maximum for that denominator which is equal to zero, which happens to be the divisor decreased to the maximum. [Assigning] a specific value would harm the largeness, since it is possible to have a higher value than that. Therefore, *khahara* has been stated [by Bhāskara] to indicate the infinite result.

This comment appears to be inspired from Kṛṣṇa Daivajña's commentary on the related verse in the *Bījagaṇita*. See [BīGa1930, p. 19].

the tip of the shadow to the base of the lamp, and (ii) the height of the lamp, even when the distance between the lamp and the *śaṅku* is not known.

छायाग्रयोरन्तरसंगुणा भा छायाप्रमाणान्तरहृद्भवेद्भूः ।
भूशङ्कुघातः प्रभया विभक्तः प्रजायते दीपशिखौच्च्यमेवम् ।
त्रैराशिकेनैव यदेतदुक्तं व्याप्तं स्वभेदैर्हरिणेव विश्वम् ॥२३९॥ । उपजातिः ।

chāyāgrayorantarasaṃguṇā bhā chāyāpramāṇāntarahṛdbhavedbhūḥ |
bhūśaṅkughātaḥ prabhayā vibhaktaḥ prajāyate dīpaśikhauccyamevam |
trairāśikenaiva yadetaduktaṃ vyāptaṃ svabhedairhariṇeva viśvam ||239||

The shadow multiplied by the difference of the tips of shadows divided by the difference of the measures of the shadows would be the [corresponding] base (*bhū*). The product of the base and the *śaṅku* divided by the [corresponding] shadow becomes the height of the flame of the lamp. Thus, whatever [mathematics] is stated is pervaded by *trairāśika* only through its variants, like the world [is pervaded] by Hari.

The setup required to determine the above results is shown in Figure 4. The method involves first placing the *śaṅku* (ST) at an arbitrary distance from the lamp and measuring the length (c_1) of the shadow. Next, the same *śaṅku* ($S'T'$) is placed further away from the lamp at a known distance (l) from its first location, and the length (c_2) of the new shadow is again measured. As can be seen from the figure, the distance between the tips of the two shadows in this case will be

$$t = c_2 + l - c_1.$$

Further, let us define the *bhū* or the base of each shadow as the distance from the tip of the shadow to the foot of the lamp. Let $CM = b_1$ be the base corresponding to the first shadow $CT = c_1$. Similarly, let $C'M = b_2$ be the base corresponding the second shadow $C'T' = c_2$. Finally, let h_s, be the known height of the *śaṅku*, and let h_l be the height of the lamp which is to be determined. Then, the verse gives the relations for the lengths of the bases and the height of the lamp as follows:

$$b_1 = \frac{c_1 t}{c_2 - c_1} \tag{9}$$

$$b_2 = \frac{c_2 t}{c_2 - c_1} \tag{10}$$

$$h_l = \frac{b_1 h_s}{c_1} = \frac{b_2 h_s}{c_2}. \tag{11}$$

In the *Vāsanābhāṣya*, Bhāskara describes the following *trairāśika* for deriving the expression for the height of the lamp [Līlā1937, p. 249]. In the given figure, in similar triangles LMC and STC, we have

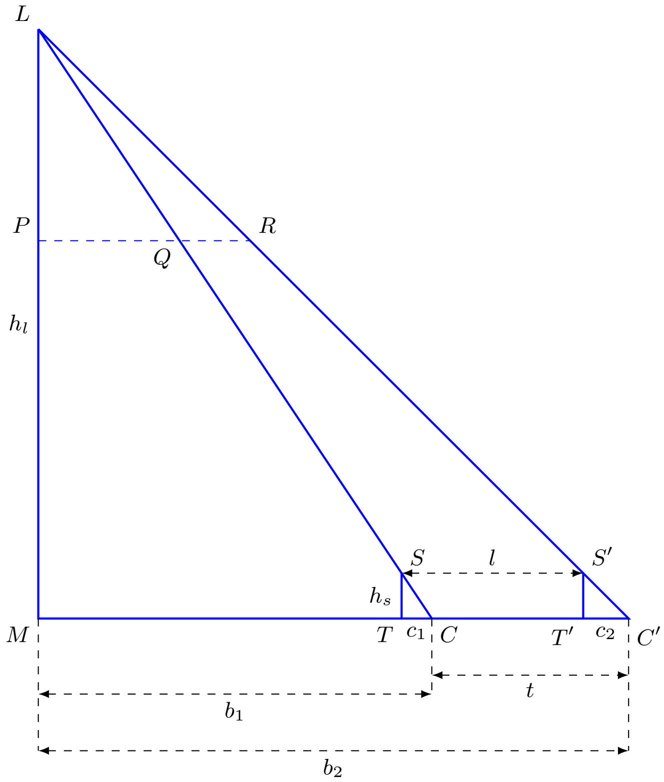

Figure 4: Determining the height of the lamp.

$$\frac{LM}{MT} = \frac{ST}{TC} \qquad \text{or,} \qquad h_l = \frac{b_1 h_s}{c_1}.$$

Similarly, from similar triangles LMC' and $S'T'C'$, we can show that

$$h_l = \frac{b_2 h_s}{c_2}.$$

Bhāskara also very tersely hints at the procedure for derivation of the expressions (9)–(10) for the bases in the *Vāsanābhāṣya* [Līlā1937, p. 249]. Bhāskara's hint can be elaborated as follows. In Figure 4, when the difference of the shadows is $c_2 - c_1$, then the distance between their tips is equal to t. If instead, the first *śaṅku* is placed at the foot of the pole upon which the lamp is perched,[28] then the shadow length (c_1) will be zero, and the difference of the lengths of the shadows will be c_2. The distance between the tips of the

[28] That is, the *śaṅku* ST is placed at M. The position of *śaṅku* $S'T'$ remains unchanged.

shadows will be equal to b_2. Then, using *trairāśika*, we have

$$\frac{b_2}{c_2} = \frac{t}{c_2 - c_1} \qquad \text{or,} \qquad b_2 = \frac{c_2 t}{c_2 - c_1}.$$

Also,

$$b_1 = b_2 - t = \frac{c_1 t}{c_2 - c_1}.$$

The derivation of the above result can also be done by considering two pairs of similar triangles (i) LPQ and STC, and (ii) LPR and $S'T'C'$, but that's far less creative than the approach presented above.

What is far more interesting is Bhāskara's pithy statement regarding the pervasiveness of *trairāśika*. This statement seems to have been made after a careful observation and study of the application of the principle of *trairāśika* to a wide array of problems in all branches of mathematics—arithmetic, algebra, and geometry. It is for this reason Bhāskara makes a proclamation that *trairāśika* pervades mathematics, much like the supreme godhead Hari (Viṣṇu or Nārāyaṇa) pervades the world, which is his creation!

7 Conclusion

Poets usually thrive on human sentiments like love, longing, nostalgia, and pride, or seek inspiration from nature to peddle their art. Therefore, tales of romance, the beauty of nature, and the exploits of heroes are topics which lend themselves naturally to a poet's imagination. Consequently, it is also quite rare for poets to put their pen to 'dry' topics like science and mathematics, which except in rare cases, fail to set the heart aflutter or the mind atwitter. It is therefore to the credit of Indian mathematicians and astronomers that they have taken up this challenge. And yet, in this pantheon of great scholars, Bhāskara stands almost alone in his ability to weave the principles of mathematics into delightful verses, without losing clarity or precision. His verses are more lucid, imaginative, joyful, and melodious compared to almost any other Indian mathematician. His use of a wide variety of metres and *alaṅkāras* is unsurpassed, as is his ability to easily relate mathematics to real life through the use of appropriate examples.

In this paper, we have attempted to bring to light this unique genius of Bhāskara by illustrating his use of various poetic techniques through examples from the *Līlāvatī*. While space considerations preclude us from showcasing any more of Bhāskara's verses, the readers may take our word that the rest of the

text too is as delightful. Thinking of the uniqueness of the *Līlāvatī* in the Indian mathematical literature, the aspects that immediately surface up are:

- the topics dealt with are quite *comprehensive* (a comparison of the topics dealt with by the *Līlāvatī* and other works clearly points to the fact that the *Līlāvatī* was *state-of-the-art* for its time).
- the text presents all the formulae in a very *concise* manner (it is hard to find any loose or redundant statement in the entire text).
- the language in which the rules and the examples are presented have an exceptional *charm* (the poetic beauty of the work makes one wonder whether this is a mathematical text or a paragon of poetry).

Indeed, we contend that the enduring charm of the *Līlāvatī* for more than eight centuries has to do with the brilliant blend of mathematical ingenuity and poetic beauty that has been consistently displayed by Bhāskara throughout the work. It was noted that Bhāskara vowed to compose the *Līlāvatī* in a technically rigorous, and yet poetically alluring manner in the very first verse. It is to Bhāskara's immense credit, that he could admirably keep up his word, and how!

The *Līlāvatī* also has important lessons for current day teachers with regards to the oft discussed topic of making mathematics more interesting. The importance of the use of interesting examples, and strong correlation of mathematics to daily life cannot be overstated. An interesting experiment that would be worth conducting is to include suitable sections of the *Līlāvatī* or other works in the educational curriculum, and to teach students various mathematical concepts through verses. This is a technique which inherently lends itself to easy memorisation of rules, and also contributes to a fun learning experience. Given the large number of mathematically challenged students that the current system has produced, this experiment should be attempted soon.

Gaṇeśa Daivajña's *upapattis* for some rules in the *Līlāvatī*

K. Ramakalyani*

1 Introduction: Gaṇeśa's date and works

Commendably consolidating the works of his predecessors and brilliantly employing his poetic skills, Bhāskarācārya composed his *Līlāvatī* (L)—a treatise that deals with arithmetic, algebra and geometry without the knowledge of which one cannot appreciate the later chapters in the *Grahagaṇita* and *Golādhyāya*. Due to the genius and poetical elegance displayed by Bhāskara, the *Līlāvatī* has become very popular compared to the earlier arithmetic works. Without the aid of a good commentary it is not easy to comprehend the intention of Bhāskara entirely. One such erudite commentary is *Buddhivilāsinī* (*BV*) of Gaṇeśa Daivajña.

Gaṇeśa Daivajña was born in 1507 CE to Keśava Daivajña, a famous astronomer, and Lakṣmī[1] at Nandigrāma on the western sea-coast. Gaṇeśa's astronomical text called the *Grahalāghava* is used in most parts of India even today by makers of traditional *pañcāṅgas*. It has been surmised that

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[1] At the end of each chapter of his *BV* Gaṇeśa records:

ज्योतिर्वित्कुलमण्डनं द्विजयतिः श्रीकेशवोऽजीजनद्
यं लक्ष्मीश्च समस्तशास्त्रनिपुणं श्रीमद्गणेशाभिधम् ।

jyotirvitkulamaṇḍanaṃ dvijayatiḥ śrīkeśavo'jījanad
yaṃ lakṣmīśca samastaśāstranipuṇaṃ śrīmadgaṇeśābhīdham |

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Gaṇeśa Daivajña wrote this work at the age of 13.[2] His other works include *Laghu-tithi-cintāmaṇi, Bṛhat-tithi-cintāmaṇi, Vivāha-vṛndāvana-ṭīkā, Muhūrta-tattva-ṭīkā, Śrāddha-nirṇaya, Parva-nirṇaya, Buddhivilāsinī* and so on [Rao2004, p. 161].

Gaṇeśa, like Bhāskara, is a poet as well as a mathematician. *BV* was written by him in 1545 CE. In this work, he gives verses with *bandhas*, quotes from various works on mathematics and astronomy such as the *Siddhāntaśiromaṇi*, the *Mahāsiddhānta* and the *Brāhmasphuṭa-siddhānta* and other texts such as the *Vākyavṛtti*, the *Vṛttaratnākara* and the *Amarakośa*; and provides grammatical notes and so on. In addition to these, *BV* is appreciated for the *upapattis* provided by Gaṇeśa, for most of the rules and examples of the *Līlāvatī*. Gaṇeśa also provides his own additional rules and examples. The explanations and *upapattis* given by *Gaṇeśa* have been very essential for the understanding of the *Līlāvatī*. Realising this, while translating the *Līlāvatī*, Colebrooke [Col1993] often adds the commentary of Gaṇeśa in the footnotes. Of late, Balacandra Rao in his introduction to the *Grahalāghava* observes, "Gaṇeśa's commentary, *Buddhivilāsinī* on the *Līlāvatī*, is an extremely useful text to understand the rationales for the formulae and methods used by Bhāskara II and his predecessors" [GrLā2006, p. ix].

Gaṇeśa provides about a hundred *upapattis* in *BV*, for the rules and examples of the *Līlāvatī*. He gives his *upapattis* in the form of

1. logical explanation,
2. algebraic proofs (using rules of the *Bījagaṇita*), and
3. demonstrations and geometric proofs.

In the following section we consider an example for each type of *upapatti* and elucidate them. The *upapattis* are chosen to represent the rules of arithmetic, algebra as well as geometry. In certain places, *BV* is compared with another major commentary, namely, *Kriyākramakarī* [Līlā1975] of Saṅkara Vāriyar and Nārāyaṇa.

2 Logical explanation

Logical explanation is a convincing and relevant argument which can be analytical and reasonable in nature. The *Līlāvatī* gives rules [Līlā1937, v. 45] for operations with zero:

[2] See [GrLā2006, p. iii]. This however seems to be purely based on the fact that the epoch chosen for his *Grahalāghava* is 1520 CE.

योगे खं क्षेपसमं ... खगुणः खं खगुणश्चिन्त्यश्च शेषविध ॥

yoge khaṃ kṣepasamaṃ ... khaguṇaḥ khaṃ khaguṇaścintyaśca śeṣavidhau ||

In addition, cipher makes the sum equal to the additive .... The product of cipher is nought; but it must be retained as a multiple of cipher, if any further operation impend [Col1993, p. 25].

Here for the rule '*khaguṇaḥ kham*' means 'a quantity multiplied by zero becomes zero'. Gaṇeśa explains this as follows [Līlā1937, p. 40]:

खगुणः खमिति त्रयेण गुणा दश त्रिंशत् स्युः। तेनैकोनेन गुणा दश विंशतिः स्युः। तेनाप्येकोनेन गुणा दश दशैव स्युः। तेनाप्येकोनेन गुणा दश शून्यमेव भाव्याः। यत एकोनगुणेन गुण्ये गुणिते गुण्यतुल्यमेवापचीयते। अत उक्तं खगुणः खमिति॥

khaguṇaḥ khamiti trayeṇa guṇā daśa triṃśat syuḥ | *tenaikonena guṇā daśa viṃśatiḥ syuḥ* | *tenāpyekonena guṇā daśa daśaiva syuḥ* | *tenāpyekonena guṇā daśa śūnyameva bhāvyāḥ* | *yata ekonaguṇena guṇye guṇite guṇyatulyamevāpacīyate* | *ata uktaṃ khaguṇaḥ khamiti* ||

[It is stated that, "a quantity] multiplied by zero is zero." Ten multiplied by three becomes thirty. Ten multiplied by one less than that will become twenty. Ten multiplied by one less than that also will become ten. Ten multiplied by one less than that must become zero. Because, when the multiplicand is multiplied by one less than the multiplier, a quantity equal to the multiplicand itself is reduced. Therefore it is said: "[a quantity] multiplied by zero becomes zero."

Using modern notation what is stated above may be written as:

$$\begin{aligned} 10 \times 3 &= 30. \\ 10 \times (3-1) &= 20. \\ 10 \times (3-1-1) &= 10. \\ 10 \times (3-1-1-1) &= 10 \times 0 = 0. \end{aligned}$$

If the multiplicand (= 10) is multiplied by one less than the multiplier (here 3), then a quantity equal to the multiplicand is reduced from the product. Thus every time one is reduced from 3, ten is reduced from the product. Therefore it is said, a quantity multiplied by zero is zero.

It may be noted that, on the above rule, Saṅkara Vāriyar in his *Kriyākramakarī* [Līlā1975, p. 91] is silent and does not give an *upapatti*. He simply states:

खगुणः खेन गुणितो राशिः पुनः स्वयं शून्यो भवति ।

khaguṇaḥ khena guṇito rāśiḥ punaḥ svayaṃ śūnyo bhavati |

'Multiplied by zero', that is, a quantity multiplied by zero, itself becomes zero.

3 Algebraic proof

At times, Gaṇeśa utilizes the rules given in *Bījagaṇita* (*BG*) for providing *upapattis*. For this reason we describe these as algebraic proofs.

As an illustration of this we consider a *varga-karma* problem (v. 62):

राश्योर्ययोः कृतिवियोगयुती निरेके मूलप्रदे प्रवद तौ मम मित्र यत्र ।
क्लिश्यन्ति बीजगणिते पटवोऽपि मूढाः षोढोक्तबीजगणितं परिभावयन्त ॥

rāśyoryayoḥ kṛtiviyogayutī nireke
mūlaprade pravada tau mama mitra yatra |
kliśyanti bījagaṇite paṭavo'pi mūḍhāḥ
ṣoḍhoktabījagaṇitaṃ paribhāvayantaḥ ||

Tell me, my friend, numbers, the sum and difference of whose squares, less one, afford square roots, which dull smatterers in algebra labour to excruciate, puzzling for it in the six-fold method of discovery there taught [Col1993, p. 35].

The problem here is to find two quantities, x and y, $(y > x)$ such that the sum of the squares of the two quantities less one $(x^2 + y^2 - 1)$ and the difference of the same two quantities less one $(y^2 - x^2 - 1)$ are square numbers.

Bhāskara gives two solutions to the above problem in two rules (vv. 60–61) as follows:

इष्टकृतिरष्टगुणिता व्येका दलिता विभाजितेष्टेन ।
एकः स्यादस्य कृतिर्दलिता सैकाऽपरो राशिः ॥
रूपं द्विगुणेष्टहृतं सेष्टं प्रथमोऽथवाऽपरो रूपम् ।
ते युतिवियुती व्येके वर्गौ स्यातां ययो राश्योः ॥

iṣṭakṛtiraṣṭaguṇitā vyekā dalitā vibhājiteṣṭena |
ekaḥ syādasya kṛtirdalitā saikā'paro rāśiḥ ||
rūpaṃ dviguṇeṣṭahṛtaṃ seṣṭaṃ prathamo'thavā'paro rūpam |
te yutiviyutī vyeke vargau syātāṃ yayo rāśyoḥ ||

The square of an arbitrary number, multiplied by eight and lessened by one, then halved and divided by the assumed number, is one quantity; its square, halved and added to one, is the other. Or unity, divided by double an assumed number and added to that number, is a first quantity; and unity is the other. These give pairs of quantities, the sum and difference of whose squares, lessened by one, are squares [Col1993, p. 35].

The pair of solutions (x, y) which satisfy the condition that

$(x^2 + y^2 - 1)$ and $(y^2 - x^2 - 1)$ are square numbers are as follows:

(i) $x = \frac{1}{2a}(8a^2 - 1)$ and $y = \dfrac{1}{2}x^2 + 1$, (for any $a > 0$)

(ii) $y = \left(\dfrac{1}{2m} + m\right)$ and $x = 1$, (for any $m > 0$)

In solution (ii) above, when $m = 1$, $y = \left(\dfrac{1}{2.1} + 1\right) = \dfrac{3}{2}$;

When $m = 2$, then $y = \left(\dfrac{1}{2.2} + 2\right) = \dfrac{9}{4}$.

Thus the solutions are $\left(1, \dfrac{3}{2}\right), \left(1, \dfrac{9}{4}\right)$ and so on.

Gaṇeśa gives *upapattis* for both the solutions. Here we consider the proof for the second solution [Līlā1937, p. 59]:

द्वितीयप्रकारस्योपपत्तिः - यथा कृतियुतिवियुतिर्व्येका मूलदा भवेत् तथा कल्पितौ राशिवर्गौ याव १ रू २ [3] । रू १। अत्र द्वितीयराशेर्मूलं रूपमेव। व्येकस्य योगस्यास्य याव १ रू २। वर्गप्रकृत्या मूलम्। तत्र `इष्टभक्तो द्विधा क्षेपः' इत्यादिना इष्टभक्तेन रूपेणेष्टेन ज्येष्ठं मूलं ३/२। अयं प्रथमो राशिः। एवमेतौ राशी ३/२। १/१। द्विकेनेष्टेन ज्येष्ठमूलं ९/४। अयं प्रथमो राशिः। एवमेतौ राशी ९। १ ।[4] एवमिष्टवशात् प्रथमराशिरनेकधा। सर्वत्र द्वितीयो रूपमेव। ४।१। अत्र द्विकतुल्या एव क्षेपे सति 'इष्टभक्तो द्विधा क्षेपः' इत्यादिना ज्येष्ठमूले क्रियमाणे तुल्यगुणहारनाशे कृते सति रूपं द्विगुणेष्टहृतमित्याद्युपपद्यते।[5]

dvitīyaprakārasyopapattiḥ - yathā kṛtiyutiviyutirvyekā mūladā bhavettathā kalpitau rāśivargau yāva 1 rū 2 | *rū 1 atra dvitīyarāśermūlaṃ rūpameva vyekasya yogasyāsya yāva 1 rū 2 vargaprakṛtyā mūlam tatra iṣṭabhakto dvidhā kṣepaḥ ityādinā iṣṭabhaktena rūpeṇeṣṭena jyeṣṭhaṃ mūlaṃ* $\frac{3}{2}$ | *ayaṃ prathamo rāśiḥ* | *evametau rāśī* $\frac{3}{2}$ | $\frac{1}{1}$ | *dvikeneṣṭena jyeṣṭhamūlaṃ* $\frac{9}{4}$ | *ayaṃ prathamo rāśiḥ* | *evametau rāśī* 9 | 1 | *evamiṣṭavaśāt prathamarāśiranekadhā* | *sarvatra dvitīyo rūpameva 4* | *1 atra dvikatulyā eva kṣepe sati iṣṭabhakto dvidhā kṣepaḥ ityādinā jyeṣṭhamūle kriyamāṇe tulyaguṇahāranāśe kṛte sati rūpaṃ dviguṇeṣṭahṛtamityādyupapadyate* |

The *upapatti* for the second method – In order that one less than the sum and difference of squares should yield square roots, the two square quantities are assumed as *yā va* 1 *rū* 2; *rū* 1. Here the square root of the second quantity '1' is one itself. One less than the sum of these is *yā va* 1 *rū* 2. The root [is found] by *vargaprakṛti* [procedure]. There, by '*iṣṭabhakto dvidhā kṣepa …*' [BīGa2006, VI.45] dividing by the assumed number 1, the greater root is $\left(\dfrac{3}{2}\right)$. This is the first quantity. Thus the two quantities are $\left(\dfrac{3}{2}, \dfrac{1}{1}\right)$. With 2 as the assumed number, the greater root is $\dfrac{9}{4}$. This is first quantity. Thus the two quantities are $\left(\dfrac{9}{4}, \dfrac{1}{1}\right)$. Thus the first

[3] It is printed as याव १ रू। १ रू १। in Apte's edition [Līlā1937, p. 59]. Only if it is taken as याव १ रू २, it satisfies the expression in व्येकस्य योगस्यास्य याव १ रू २। Also, see [Ms-2279, leaf. 20, line 14], which gives this as याव १ रू २ रू १।

[4] [Līlā1937, p. 60] The denominators are misplaced in the text.

[5] The reading in printed text here is द्विगुण [Līlā1937, p. 60]. It must be द्विगुणेष्ट.

quantity can be many for different desired numbers. Everywhere the second is 1 only. Here when the addendum is equal to 2, then while computing the greater root by '*iṣṭabhakto dvidhā kṣepa ...*', by cancelling the equal multiplier and divisor, "unity, divided by double an assumed number..." ('*rūpaṃ dviguṇeṣṭahṛtaṃ...*' L.61) is obtained.

If x and y are two quantities then $(x^2 + y^2 - 1)$ and $(-x^2 + y^2 - 1)$ must yield square roots. Let the two square quantities y^2 and x^2 be respectively $(a^2 + 2)$ and 1. The root of the second quantity 1 is 1 itself.

One less than difference of these quantities $(-x^2+y^2-1) = -1+a^2+2-1 = a^2$. This is a square by itself.

One less than sum of these quantities $(x^2+y^2-1) = 1+a^2+2-1 = a^2+2$.

Since this has to be a square, let this be equated to 'b^2', that is, $a^2+2 = b^2$.

The roots of this can be found by the rule found in the *Bījagaṇita*, "*iṣṭabhakto...*" [BīGa2006, VI.45]:

इष्टभक्तो द्विधा क्षेप इष्टोनाढ्यो दलीकृतः ।
गुणमूलहृतः चाऽद्यो ह्रस्वज्येष्ठे क्रमात् पदे ॥

iṣṭabhakto dvidhā kṣepa iṣṭonāḍhyo dalīkṛtaḥ |
guṇamūlahṛtaḥ cā'dyo hṛsvajyeṣṭhe kramāt pade ||

The additive, divided by an assumed quantity, is twice set down, and the assumed quantity is subtracted in one instance, and added in the other; each is halved; and the first is divided by the square-root of the multiplier [that is, coefficient.] The results are the 'least' and 'greatest' roots in their order [Col2005, p. 182].

In the equation, $a^2 + 2 = b^2$, the coefficient of *prakṛti* is 1, a perfect square. Let the optional number be 1. The additive is to be divided by the optional number 1. That is,

$$\frac{2}{1} = 2; \text{ then } a = \frac{2-1}{2\sqrt{1}} \text{ and } b = \frac{2+1}{2} = \frac{3}{2}.$$

Here the *jyeṣṭhamūla* is $\dfrac{3}{2}$; the other *mūla* is 1 itself.

When the optional number is $\frac{1}{2}$, then

$$\frac{2}{\frac{1}{2}} = 4; \text{ then } a = \frac{4-\frac{1}{2}}{2\sqrt{1}} = \frac{7}{4} \text{ and } b = \frac{4+\frac{1}{2}}{2} = \frac{9}{4}.$$

Here the *jyeṣṭhamūla* is $\dfrac{9}{4}$. Thus by choosing different suitable optional numbers, many values can be found for one quantity and the other quantity is always 1. Thus the solutions are $\left(1, \dfrac{3}{2}\right), \left(1, \dfrac{9}{4}\right)$ and so on.

Here when the addendum (*kṣepa*) is equal to 2, while creating the greater root, by the rule, "*iṣṭabhakto …*", by cancelling the equal multiplier and divisor,"*rūpaṃ dviguṇeṣṭahṛtaṃ…*' is obtained.

In the equation, $a^2 + 2 = b^2$, the addendum is 2.

By the rule, "*iṣṭabhakto …*", by assuming optional number as 'n', the smaller root (a) and the greater root (b) are given by

$$a = \frac{\left(\frac{2}{n} - n\right)}{2\sqrt{1}} \quad \text{and} \quad b = \frac{\left(\frac{2}{n} + n\right)}{2} = \frac{2}{n} \times \frac{1}{2} + \frac{n}{2}.$$

By cancelling 2, the equal multiplier and the divisor in the first term, and (substituting $2m$ for n) we get

$$b = \frac{1}{n} + \frac{n}{2} = \frac{1}{2m} + m.$$

This explains Bhāskara's rule,'*rūpaṃ dviguṇeṣṭahṛtaṃ…*' [Līlā1937, pp. 57–58]. In this context, Gaṇeśa gives his own rule [Līlā1937, p. 60]:

> अनयैव युक्त्या प्रकारान्तरं मया निबद्धं `इष्टकृतिर्यमयुक्ता द्विगुणेष्टहृताऽथवापरो रूपं' इति ।
>
> *anayaiva yuktyā prakārāntaraṃ mayā nibaddhaṃ 'iṣṭakṛtiryamayuktā dviguṇeṣṭahṛtā'thavāparo rūpaṃ' iti* |
>
> By this rationale only, another method is formulated by me as follows: "Or, otherwise, the square of the chosen number added to two and divided by twice the chosen number [is one quantity] and the other is one.

What is stated here is that the greater root b which was given by

$$b = \frac{\left(\frac{2}{n} + n\right)}{2},$$

can also be written as

$$b = \left(\frac{2 + n^2}{2n}\right).$$

4 Demonstration and geometric proof

Bhāskara gives a rule for finding the area of a circle and finding the surface area and volume of a sphere [Līlā1937, v. 201]:

> वृत्तक्षेत्रे परिधिगुणितव्यासपादः फलं यत्
> क्षुण्णं वेदैरुपरि परितः कन्दुकस्येव जालम् ।

गोलस्यैवं तदपि च फलं पृष्ठजं व्यासनिघ्नं
षड्भिर्भक्तं भवति नियतं गोलगर्भे घनाख्यम् ॥

vṛttakṣetre paridhiguṇitavyāsapādaḥ phalaṃ yat
kṣuṇṇaṃ vedairupari paritaḥ kandukasyeva jālam |
golasyaivaṃ tadapi ca phalaṃ pṛṣṭhajaṃ vyāsanighnaṃ
ṣaḍbhirbhaktaṃ bhavati niyataṃ golagarbhe ghanākhyam ||

In a circle, a quarter of the diameter multiplied by the circumference is the area. That multiplied by four is the net all around the ball. This surface area of the sphere, multiplied by diameter and divided by six, is the precise solid, termed cubic, content within the sphere [Col1993, p. 88].

If d and c denote the diameter and the circumference of a circle, then the formulae given in the above verses for the area of the circle (A), the surface area of the sphere (S), and the volume of the sphere (V) may be expressed as:

$$A = \frac{1}{4}d \times c = \frac{1}{4} \text{ of diameter } (\textit{vyāsa}) \times \text{circumference } (\textit{paridhi}),$$
$$S = \left(\frac{1}{4}d \times c\right) \times 4 = \text{area of the circle} \times 4,$$

$$\begin{aligned} V &= \text{the surface area} \times \text{diameter} \div 6 \\ &= \left(\frac{1}{4}d \times c\right) \times 4 \times \frac{d}{6}. \end{aligned}$$

4.1 Area of a circle

For the expression given above for the area of a circle, Gaṇeśa gives the following *upapatti* [Līlā1937, p. 200]:

अत्रोपपत्तिः – वृत्तक्षेत्रस्य समं खण्डद्वयं कृत्वा तयोः खण्डयोर्यथेप्सितानि सूच्यग्राणि शकलानि यथा स्युः तथा खण्डे छित्वा प्रसारयेत्। दर्शनम्। एते परस्परमध्ये संक्रमय्याऽऽयतं जायते तस्य दर्शनम्। अत्र व्यासार्धमेको भुजः। परिध्यर्धमन्यः। तयोर्घातः फलम्। तदर्धमर्धेन गुणितं चतुर्थांशः स्यात्। अतः परिधिगुणितव्यासपादः समवृत्तक्षेत्रफलं स्यात् इत्युपपन्नम्॥

atropapattiḥ – vṛttakṣetrasya samaṃ khaṇḍadvayaṃ kṛtvā tayoḥ khaṇḍayoryathepsitāni sūcyagrāṇi śakalāni yathā syuḥ tathā khaṇḍe chitvā prasārayet | darśanam | ete parasparamadhye saṃkramayyā"yataṃ jāyate tasya darśanam | atra vyāsārdhameko bhujaḥ | paridhyardhamanyaḥ | tayorghātaḥ phalaṃ | tadardhamardhena guṇitaṃ caturthāṃśaḥ syāt | ataḥ paridhiguṇitavyāsapādaḥ samavṛttakṣetraphalaṃ syāt ityupapannam ||

> The *upapatti* here – Making the circular field into two equal halves, and cutting those two parts into as many as desired number of needle-shaped parts, spread them. [Here is the] demonstration. By inserting these into one another, a rectangle is produced. [Further] its demonstration is as follows: Here one side is half the diameter. The other is half the circumference. The area is their product. Half multiplied by half will become one-fourth. Hence one-fourth of the diameter multiplied by the circumference will become the area of the circle – thus [is] the demonstration.

Here, Gaṇeśa asks us first to cut the circle into two equal halves. These are then divided into needle-shaped (*sūcyagrāṇi*) sections (into as many pieces as possible) and arranged in such a way that a rectangle (*āyatam*) is formed. This is shown in Figure 1. Here it may be noted that, the bases of needle-shaped sections (sectors) are basically arcs of the circumference of the circle, but when the number of sections is large, they form nearly a straight line.

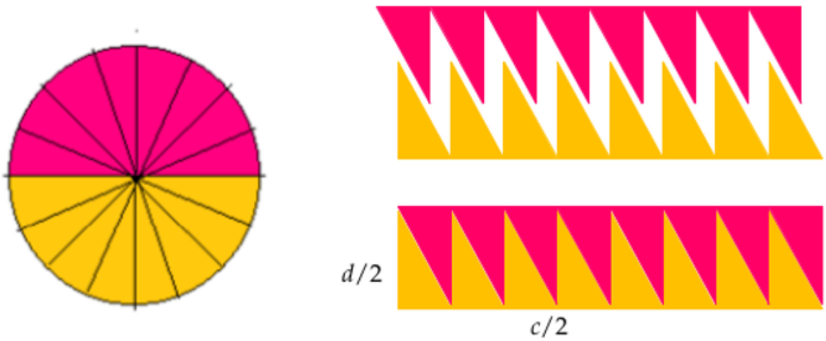

Figure 1: Dissecting a circular section into a rectangle.

In this rectangle, one side is half the diameter and other side is half the circumference. Thus, the area of circle is equal to the area of rectangle formed which is equal to the product of sides that is,

$$= \frac{1}{2}d \times \frac{1}{2}c = \frac{1}{4}d \times c.$$

In modern notation this is equal to

$$\left(\frac{1}{2}\right) 2r \times \left(\frac{1}{2}\right) 2\pi r = \pi r^2.$$

4.2 Surface area of a sphere

In order to demonstrate the expression for the surface area of a sphere, Gaṇeśa presents the following *upapatti* [Līlā1937, pp. 200–201]:

> अथ गोलपृष्ठफलोत्पत्तिः -- गोलस्य परिध्यर्धेन तुल्यव्यासं वर्तुलं वस्त्रादि छित्वा तेन वर्तुलवस्त्रेण स गोलोऽर्धश्चेदाच्छाद्यते। तद्वस्त्रं नीवीसदृशं किञ्चिच्छिष्यते। अतो न्यूनं तद्वस्त्रफलमर्धगोलस्य पृष्ठफलं स्यात्। वस्त्रफलं तु गोलस्य परिधिगुणितव्यासपादस्य सार्धद्विगुणितस्याऽऽसन्नं भवेत्। अतो वस्त्रस्य किञ्चिदवशिष्टत्वाद्गोलस्य परिधिगुणितव्यासपादो द्विगुणित एव गोलार्धे पृष्ठफलं कल्प्यते। अतश्चतुर्गुणितः परिधिगुणितव्यासपादः गोलपृष्ठफलं स्यात्। अत उक्तं - `फलं तत्क्षुण्णं वेदै' रित्यादि।
>
> *atha golapṛṣṭhaphalotpattiḥ – golasya paridhyardhena tulyavyāsaṃ vartulaṃ vastrādicchitvā tena vartulavastreṇa sa golo'rdhaścedācchādyate | tadvastraṃ nīvīsadṛśaṃ kiṃcicchiṣyate | ato nyūnaṃ tadvastraphalamardhagolasya pṛṣṭhaphalaṃ syāt | vastraphalaṃ tu golasya paridhiguṇitavyāsapādasya sārdhadviguṇitasyā''sannaṃ bhavet | ato vastrasya kiṃcidavaśiṣṭatvādgolasya paridhiguṇitavyāsapādo dviguṇita eva golārdhe pṛṣṭhaphalaṃ kalpyate | ataścaturguṇitaḥ paridhiguṇitavyāsapādaḥ golapṛṣṭhaphalaṃ syāt | ata uktaṃ - 'phalaṃ tatkṣuṇṇaṃ vedair' ityādi |*
>
> Then the *upapatti* for the surface area of a sphere – From cloth etc., cutting a circle whose diameter is equal to half of the circumference of the sphere, if by that circular cloth that half sphere is covered, that cloth remains a little like pleats [of women's garments]. Hence a little less than the area of that cloth will be the surface area of the half sphere. The area of the [circular] cloth will be nearly two and a half times the quarter of the diameter multiplied by the circumference of the sphere. Therefore, as a little cloth is remaining, two times the quarter of the diameter multiplied by the circumference of the sphere is considered as the surface area of half the sphere. So four times the quarter of the diameter multiplied by the circumference of the sphere will be the surface area of the sphere. Therefore it is said thus – 'area, that multiplied by four'.

The demonstration given by Gaṇeśa can be explained as follows: A circular cloth whose diameter is equal to half of the circumference of the sphere is taken. Half the sphere is covered by this circular cloth. After covering the hemisphere (*golārdha*), a small piece of cloth just like the pleats of a garment is left. When this extra cloth is subtracted from the circular cloth, the remaining gives the surface area of hemisphere.

Let C, D and R be respectively the circumference, diameter and radius of the circular cloth. Let c, d and r be respectively the circumference of the great circle, diameter and radius of the sphere. Now, the area of the circular cloth is

$$A_c = \frac{1}{4} D \times C \sim \left(\frac{5}{2}\right) \frac{1}{4} d \times c.$$

The area of hemisphere

$$\begin{aligned} &= A_c - \text{area of remaining cloth}^6 \\ &= 2\left(\frac{1}{4}d \times c\right). \end{aligned}$$

Thus, the surface area of the sphere is

$$S = 2 \times 2\left(\frac{1}{4}d \times c\right) = 4\left(\frac{1}{4}d \times c\right) = d \times c.$$

Using modern notation for π, this can be written as

$$4\left(\frac{1}{4} \times 2r \times 2\pi r\right) = 4\pi r^2.$$

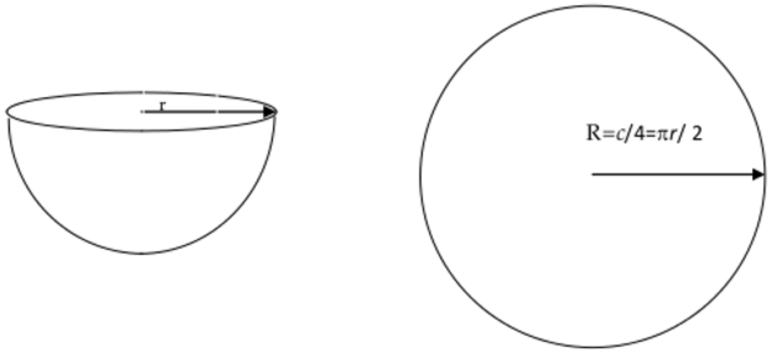

Figure 2: The hemisphere and circular cloth taken to demonstrate the surface area of sphere.

Note 1: Gaṇeśa probably has taken the idea for this *upapatti* from *Golādhyāya* [SiŚi1943, vv. 52–56, p. 82] of Bhāskara. A similar explanation is found in *Kriyākramakarī* also [Līlā1975, vv. 17–19, p. 397].

Note 2: The demonstration shows that the area of the cloth taken is little more than the surface area of the hemisphere. Gaṇeśa does not mention how $\left(\frac{5}{2}\right)\left(\frac{1}{4}d \times c\right)$ is obtained. [This can be obtained if one takes π^2 to be 10:

[6] It may be mentioned here that Gaṇeśa does not explain how to obtain the area of the remaining cloth that constitutes the pleats.

Area of cloth is $\frac{1}{4}D\times C = \frac{1}{4}\pi r\times\pi(\pi r) = \frac{1}{4}\pi r^2\pi^2 \sim \frac{1}{4}\pi r^2\times 10 = \left(\frac{5}{2}\right)\frac{1}{4}d\times c$]. The extra piece of cloth left like *nīvī* after covering might have been actually cut and measured to see[7] that it was equal to $\left(\frac{1}{2}\right)\frac{1}{4}d\times c$.

5 Volume of sphere

Gaṇeśa gives the following *upapatti* for the volume of a sphere [Līlā1937, p. 201]:

> अथ घनफलोपपत्तिः – गोलस्य व्यासार्धतुल्यदैर्घ्याणि सूच्याकाराणि सूच्यग्राणि चतुष्कोण-दैर्घ्याणि मूर्ध्नि हस्ततुल्यविस्तृतिदैर्घ्याणि कृतानि खण्डानि पृष्ठफलसंख्यान्येव भवन्ति। एवं विधैकखण्डस्य मूर्ध्नि क्षेत्रफलं रूपमेव। खण्डदैर्घ्यं स एव वेधः। तेन गुणितं क्षेत्रफलं तस्य त्र्यंशो घनफलं स्यात्। ``क्षेत्रफलं वेधगुणं खाते घनहस्तसंख्या स्यात्"।[8] ``समखातफलत्र्यंशः सूचीखाते फलं भवती"ति[9] वक्ष्यमाणत्वात्। अतो व्यासषडंश एवैकखण्डस्य घनफलं स्यात्। तत्पृष्ठफलगुणितं सर्वगोलस्य घनफलं जायत इति। अत उक्तं - `तदपि च फलं पृष्ठज' मित्यादि॥
>
> *atha ghanaphalopapattiḥ - golasya vyāsārdhatulyadairghyāṇi sūcyākārāṇi sūcyagrāṇi catuṣkoṇadairghyāṇi mūrdhni hastatulyavistṛtidairghyāṇi kṛtāni khaṇḍāni pṛṣṭhaphalasaṃkhyānyeva bhavanti | evaṃ vidhaikakhaṇḍasya mūrdhni kṣetraphalaṃ rūpameva | khaṇḍadairghyaṃ sa eva vedhaḥ | tena guṇitaṃ kṣetraphalaṃ tasya tryaṃśo ghanaphalaṃ syāt | "kṣetraphalaṃ vedhaguṇaṃ khāte ghanahastasaṃkhyā syāt" | "samakhātaphalatryaṃśaḥ sūcīkhāte phalaṃ bhavatīti" vakṣyamāṇatvāt | ato vyāsaṣaḍaṃśa evaikakhaṇḍasya ghanaphalaṃ syāt | tatpṛṣṭhaphalaguṇitaṃ sarvagolasya ghanaphalaṃ jāyata iti | ata uktaṃ - 'tadapi ca phalaṃ pṛṣṭhajam' ityādi ||*
>
> Then the *upapatti* for cubical content – Needle-shaped pointed sections (pyramids), which have length equal to half the diameter of the sphere and which have on top, squares that have length and width equal to one *hasta* (unit length), are made and the number of such sections are equal to the surface area. The area on top of such a section is one itself. The length of a section is half the diameter, that itself is depth. One third of the area multiplied with that will be the cubical content, as it

[7] An experiment was actually carried out by the author of this paper to verify Gaṇeśa's statement. A hollow hemisphere was taken. The circumference (c) of its great circle was measured using a thread, which was found to be 42cm. First a circular paper of diameter (D) 21cm $\left(=\frac{c}{2}\right)$ was cut and the demonstration could not be carried out as the paper could not cover the hemisphere properly due to lack of flexibility. Then a thin circular cloth whose diameter = 21cm was cut. This was spread over the hemisphere to cover it properly and it was glued to the surface of the hemisphere (to prevent movement). The remaining extra cloth was cut into small pieces and arranged on a plane squared sheet marked with squared of area 1 sq.cm. The area of these remaining cloth pieces put together was nearly equal to 59 sq.cm, which was a little less than $\frac{1}{2}\times\frac{1}{4}d\times c = \frac{1}{2}\times\frac{1}{4}\times\frac{42}{22}\times 7\times 42 = 70.159$ sq.cm.

[8] [Līlā1937, v. 214, p. 220].

[9] [Līlā1937, v. 217, p. 223].

> will be said that 'the area multiplied by the depth will be the number of cubical *hastas* of an excavation' and that 'one third of the [cubical] content of a regular excavation is the [cubical] content of a needle-shaped excavation'. Therefore, one sixth of the diameter only is the cubical content of one section. That multiplied by the surface area becomes the cubical content of the full sphere. Hence it is said – 'this surface area also ...'

The whole surface area of the sphere is divided into unit squares whose side is equal to one *hasta*. Let the total area of the sphere be made up of S such unit squares. Corresponding to each unit square, one needle-shaped pyramid (*sūcyagra*) is conceived of. The base area of each needle- shaped pyramid is obviously a square of unit length. It is further stated that the depth of the conceived pyramid is equal to half of the diameter. The *upapatti* given above may be explained as follows:

Hence the volume of one such pyramid is given by

$$V_{pyramid} = \text{base area} \times \text{depth} \times \frac{1}{3}.$$

Thus, the volume of the sphere is

$$V_{sphere} = S \times V_{pyramid} = S \times \frac{d}{6}.$$

This can be expressed using modern notation as

$$= 4\pi r^2 \times \left(\frac{2r}{6}\right) = \left(\frac{4\pi r^2}{3}\right) \text{ cubic units.}$$

6 Conclusion

Based on our study, we would like to mention that the *Buddhivilāsinī* is one of the earliest commentaries that presents *upapattis* to various rules given by Bhāskara in his *Līlāvatī*. Gaṇeśa Daivajña commences his commentary with the following interesting proclamation. [Līlā1937, p. 1]:

> अत्रोपपत्तिकथनेऽखिलसारभूते पश्यन्तु सुज्ञगणका मम बुद्धिचित्रम् ।
>
> *atropapattikathane 'khilasārabhūte paśyantu sujñagaṇakā mama buddhicitram* |
>
> May expert mathematicians observe the wonder of my intelligence in narrating the *upapattis* that captures the essence of a variety of aspects [of mathematics].

A variety of *upapattis* presented in the text indeed demonstrates that this statement has been upheld by him.

A study of two Malayalam commentaries on the *Līlāvatī*

N. K. Sundareswaran*[*] and P. M. Vrinda

1 Introduction

The doyen of scholars Dr. K. V. Sarma, who completely dedicated himself to unearth, identify and edit seminal works produced by the mathematical tradition of Kerala, has recorded that there are six commentaries written in Malayalam on the *Līlāvatī* [Līlā1975, p. xiii]. He has given a detailed list of manuscripts of these commentaries in his work *A History of the Kerala School of Hindu Astronomy* [Sar1972, pp. 162–163].[1] Many of these manuscripts are housed in the Oriental Research Institute and Manuscripts Library, Trivandrum, and the rest are from private collections. With a view to assess the scope and feasibility of editing any/some of these commentaries, we made a fresh survey of manuscripts of Malayalam commentaries on *Līlāvatī* deposited in the three public repositories of manuscripts in Kerala viz.

1. Oriental Research Institute and Manuscripts Library, Trivandrum,
2. Sree Ramavarma Government Sanskrit College Grantha Library, Trippunithura, and
3. Thunchan Manuscripts Repository, University of Calicut.

Our original proposal was to have a close study of all the manuscripts of commentaries written in the Malayalam language on the *Līlāvatī*. But as we could not succeed in getting physical access to all the manuscripts, we are constrained to confine our study to the manuscripts of two commentaries. In

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[1] Besides this, Sarma has also prepared a catalogue giving details of scientific texts in Sanskrit in various repositories of Kerala and Tamilnadu [Sar2002].

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fact, the majority of available manuscripts are of these two commentaries. In addition, these carry much significance from the point of view of the textual study of the *Līlāvatī*.

2 Details of manuscripts

In preparing this paper, we have utilized the following manuscripts:

Sl. No.	Name of the Library	Manuscript No.
1	Oriental Research Institute and Manuscripts Library, Trivandrum	17683
2	”	18438
3	”	22199
4	”	22447
5	Sri Ramavarma Government Sanskrit College Grantha Library, Trippunithura	619A
6	”	619B
7	”	620A
8	”	620B
9	Thunchan Manuscripts Repository, University of Calicut	3585A

Of these, the first four and the last one contain one commentary that we refer to as *Yogāśraya*. The remaining ones pertain to the other commentary which we refer to as *Abhipreta*. The authors of both these commentaries are unknown. Furthermore, we also do not get any specific name given by the authors themselves to refer to these commentaries. However, since the first commentary has the benedictory verse:

> योगाश्रयं पापवियोगहेतुकं गुणालयं हारमशेषदुष्कृताम् ।
> कृतीश्वरं वेदगिरां परं पदं घनत्विषं गोपकुमारमाश्रये ॥
>
> *yogāśrayaṃ pāpaviyogahetukaṃ guṇālayaṃ hāramaśeṣaduṣkṛtām* |
> *kṛtīśvaraṃ vedagirāṃ paraṃ padaṃ ghanatviṣaṃ gopakumāramāśraye* ||
>
> I seek refuge upon the Gopakumāra (Lord Kṛṣṇa), who is the object of the *yogasādhanās* [like *dhāraṇā and dhyāna*], the one who causes severance of sins, the seat of all virtues, the cleanser of wrong doers, the lord of realized souls, the ultimate one who is the subject of *upaniṣadic* passages [such as *tattvamasi*], the one whose complexion is akin to that of the rain bearing clouds.[2]

[2] It may be noted that by the terms *yoga, viyoga, guṇa, hāra, kṛti, pada*, and *ghana*, the fundamental operations addition, subtraction, multiplication, division, squaring, square-rooting and cubing are hinted at by *śabdaśaktimūladhvani*.

which commences with the word *Yogāśraya*, we have chosen the name *Yogāśraya* to refer to it. The second commentary has the following benedictory verse:

> अभिप्रेतार्थसिद्ध्यर्थं पूजितो यस्सुरैरपि ।
> सर्वविघ्नच्छिदे तस्मै गणाधिपतये नमः ॥
>
> *abhipretārthasiddhyartham pūjito yassurairapi* |
> *sarvavighnacchide tasmai gaṇādhipataye namaḥ* ||
>
> The one who is worshipped even by the Gods in order to get their undertaking accomplished, unto that Gaṇeśa who severes all difficulties [I offer] my venerations.

Since this commences with the word *'Abhipreta'*, we refer to this commentary hereafter as *Abhipreta.*

Palaeographic evidence from these manuscripts suggests that these are not later than 1850.[3] This is corroborated by the statement found in the colophon of manuscript no. 17683 where in we find the date of completion of writing as the 3rd of the month *ciṇṇam* (Malayalam for *Simha*) Malayalam Era 980. This date falls in the year 1805 CE. Here we also find the name of the scribe - *Vaḷappil Ceṟiya Rāman.*

3 Salient features of the two commentaries

In what follows we present some of the salient features of the two commentaries *Yogāśraya* and *Abhipreta.* These manuscripts essentially present brief explanations for the verses in the *Līlāvatī* in simple language. However, there are some distinct features as well.

The *Yogāśraya* commentary

The *Yogāśraya* commentary is a simple and clear exposition of the text. While commenting on the text, this commentary quotes relevant passage of the original text verbatim. The *karaṇasūtras* are mostly explained with the help of illustrative examples of the text. The *karaṇasūtras* and *udāharaṇas* (*uddeśakas*) are quoted together at first and then the *karaṇasūtras* are illustrated with the examples given in the *uddeśakas.* At times the explanatory passages from the *vāsanā* portion of the original text are reproduced without mentioning the source. Numbers are expressed in words. Sometimes they are reexpressed (i.e., duplicated) using digital symbols in Malayalam. There are no figures drawn.

[3] Thanks are due to the eminent palaeographer Prof. M. R. Raghava Variyar, who, after examining the manuscripts, gave this considered opinion.

The order of the verses (in the *Līlāvatī*) are not in conformity with that of the *Kriyākramakarī* (*KKK*) commentary. Rather they are closer in the order of the verses found in the *Buddhivilāsinī* (*BV*) commentary of Gaṇeśa Daivajña, as are many of the readings.

The section on *chandaścityādi* occurs at the end of the *miśravyavahāra*[4] in this commentary. This agrees with the order of the *BV*. But in the *KKK*, the section on *chandaścityādi* comes after the section on *śreḍhī*. Besides, the *karaṇasūtra 'pādākṣaramite gacche'* etc. appears at the end of the section (on *chandaścityādi*) in this commentary unlike in the *KKK*, where it appears at the beginning. As we will shortly see, the *Abhipreta* also agrees with the *Yogāśraya* in these respects. But this commentary does not simply follow the reading of the *BV* tradition. There are two verses in this commentary,[5] which are not seen in the *BV*, for finding out the radius of a circle from the four sides of a quadrilateral inscribed in it. These are:

दोष्णां द्वयोर्द्वयोर्घातयुतीनां तिसृणां वधे ।
एकैकोनेतरत्रैक्यचतुष्केण विभाजिते ॥
लब्धमूलेन यद्वृत्तं विष्कम्भार्धेन निर्मितम् ।
सर्वं चतुर्भुजं क्षेत्रं तस्मिन्नेवावतिष्ठते ॥

doṣṇāṃ dvayor dvayor ghātayutīnāṃ tisṝṇāṃ vadhe |
ekaikonetaratraikyacatuṣkeṇa vibhājite ||
labdhamūlena yadvṛttaṃ viṣkambhārdhena nirmitam |
sarvaṃ caturbhujaṃ kṣetraṃ tasminnevāvatiṣṭhate ||

The three sums of the products of the sides taken two at a time are to be multiplied together and divided by the product of the four sums of the sides taken three at a time and diminished by the fourth. If a circle is drawn with the square root of this quantity as radius, the whole quadrilateral will be situated on it. [Sar1999, p. 108].

The above verse essentially presents the following expression for the radius of the circle in terms of the sides of a cyclic quadrilateral. That is,

$$\text{radius} = \sqrt{\frac{(ab+cd)(ac+bd)(ad+bc)}{(a+b+c-d)(b+c+d-a)(c+d+a-d)(d+a+b-c)}}$$

where a, b, c, d are the sides.

The *KKK* also has these verses.[6] K. V. Sarma, while editing, puts these under the section on *vṛttacaturaśra* (these two verses are the sole members in this). It may be noted that here again, the position of these verses differ. In

[4] K. V. Sarma writes [Līlā1975, p. 238]: 'In the north Indian recensions of *Līlavatī*, this section on *śreḍhī* occurs after *chandaścityādi* which is treated there as the last portion of *Miśravyavahāra* recension.'

[5] These find a place in the *Abhipreta* commentary as well.

[6] There are some variations in the readings. See *Kriyākramakarī* [Līlā1975, p. 363].

the *Yogāśraya*, these come after the section on *sūcīkṣetra* while in the *KKK*, these are placed before it.[7]

There is a verse in Sanskrit which gives the cubes of counting numbers 1 to 9, using the *kaṭapayādi* system of notation. It runs as:

> यज्ञेन दानेन सुखेन वर्तनं मुरस्य तापोऽत्र गवाङ्गरोपणम् ।
> धरास्थ एकादिनवावसायिनां इत्यङ्ककानां घनमत्र विद्यताम् ॥
>
> *yajñena dānena sukhena varttanaṃ murasya tāpo'tra gavāṅgaropaṇam |*
> *dharāstha ekādinavāvasāyināṃ ityaṅkakānāṃ ghanamatra vidyatām ||*
>
> May you understand that 1 (*yajñena*), 8 (*dānena*), 27 (*sukhena*), 64 (*varttanaṃ*), 125 (*murasya*), 216 (*tāpo'tra*), 343 (*gavāṅga*), 512 (*ropaṇam*) and 729 (*dharāstha*): are the cubes of numbers from 1 to 9.

For finding the square root and cube root, an easy method is given after explaining the method enunciated in the text. We get a Sanskrit verse in the beginning with the introductory remark '*varggaghanaṅṅaḷuṭe mūlattinnu oru upadeśaślōkatte paṟayunnu*' (for the square root and the cube root, a verse is enunciated). The verse runs as:

> इष्टाप्तमिष्टान्वितमर्धितं यत् वर्गस्य मूलं घनमूलमेवम् ।
> द्विराहृतेष्टैक्यदलं विशेषे ।
>
> *iṣṭāptamiṣṭānvitamardhitaṃ yat iṣṭavargasya mūlaṃ ghanamūlamevam |*
> *dvirāhṛteṣṭaikyadalaṃ viśeṣe |*[8]

The *Abhipreta* commentary

This also is a simple and brief exposition of the text. The original text of *Līlāvatī* is never quoted in the commentary portion. Only the introductory and concluding remarks like *iti trairāśikam, iṅṅane ghanamūlam* and *atha saptarāśikam* are seen.

The commentary presupposes the text, which it seems, is given separately in the first part. Thus the manuscripts 619A and 620A contain the text alone and part B of each one contains the commentary. Here also numbers are expressed in words. Sometimes they are expressed using digital symbols in Malayalam. Unlike the *Yogāśraya*, here some geometrical figures are drawn to illustrate the point at hand.

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[7] In this aspect also our two commentaries agree with each other.

[8] It is noticed that the verse is incomplete and that, as such, it is difficult to construe. The following emendations have been made. In the first half of the verse, the reading यत् वर्गस्य was यदिष्टवर्गस्य । Similarly दलं विशेषे was फलाविशेषे । With these emendations not only is the verse metrically correct, but it also gives the same rules as those given in the Malayalam passages appearing in the next section.

Though the text more or less follows the order of the *KKK*, there are some exceptions, as in the case of the *karaṇasūtra 'pādākṣaramite gacche'* etc. It appears at the end of the section on '*chandaścityādi*' in this commentary unlike in the *KKK*, where it appears at the beginning. Just as in the case of *Yogāśraya*, the section on '*chandaścityādi*' occurs at the end of *miśravyavahāra*. In the case of '*doṣṇāṃ dvayor dvayor ghātayutīnaṃ*', the positioning is different. Here also this commentary agrees with *Yogāśraya*.

Passages from the *vāsanā* are also sometiomes explained just as the verses of the text. It is not specified that these are from the expository notes of the author, i.e. Bhāskarācārya. Many verses that are cited in the *KKK* commentary are also reproduced. It is to be noted that there is no indication either to the sources of these verses or to the fact that these do not belong to the original text. The verses of the *Līlāvatī*, the passages of the *vāsanā* appearing sporadically, and these verses are given as a running matter without any kind of demarking expression (in the first part, as is specified earlier).

There are some verses in this commentary which are not found in the *KKK* or the *BV*. For instance, for finding out the area of a circle and the volume of a sphere, it gives a simple verse in addition to the *karaṇasūtras* '*vṛttakṣetre*' etc. and '*vyāsasya varge*' etc. It runs as:

> व्यासस्य वर्गाच्छरसायकाग्निक्षुण्णाद् द्विपञ्चाब्धिहृतं फलं स्यात् ।
> घनीकृतव्यासक[9] एव नागाद्रयङ्गाहृतं गोळफलं घनाख्यम् ॥
>
> *vyāsasya vargāccharasāyakāgnikṣuṇṇād dvipañcābdhihṛtaṃ phalaṃ syāt |*
> *ghanīkṛtavyāsaka eva nāgādryaṅgāhṛtaṃ goḷaphalaṃ ghanākhyam ||*
>
> The area of a circle is got by multiplying the square of its diameter with the number 355 and dividing the product by 452. The volume of the sphere (*golaphala*) called *ghana* is [obtained by multiplying] the cube of the diameters [with 355] and dividing [the result] by 678.

It may be noted that this verse gives a better approximation for π than the value given by the *Līlāvatī* verse '*vyāse bhanandāgnihate* etc.' (3.1416) since the value obtained from the present verse amounts to taking π as $\frac{355}{113}$ which is equal to 3.1415929204.[10]

We get two more verses which do not appear in the *KKK*. They are:

> त्रयोदशभिरेकाग्रो यो राशिर्गणकोत्तम ।
> चतुस्त्रिंशोद्धृतो व्याप्तं तं राशिं वद पृच्छतः ॥

[9] Here we have emended the reading from व्यासक to व्यासत in order to have an oblative (*pañcamī*) just like वर्गात् in the first quarter of the verse.

[10] This is one among the many approximations for π given in the Kerala mathematical tradition. The *KKK* gives the same value by suggesting a variant reading for the hemistich '*vyāse bhanandāgnihate* etc.' as '*vyāse śareṣvagnihate vibhakte rāmendurūpaiḥ paridhiḥ susūkṣmaḥ*' [Līlā1975, p. 377].

द्व्याद्यैः षळ्पर्यन्तैरेकाग्रः को भवेद्राशिः ।
सप्तभिरेव स शुद्धो वद शीघ्रं गणक राशिं तम् ॥

trayodaśabhirekāgro yo rāśir gaṇakottama |
catustriṃśoddhṛto vyāptaṃ taṃ rāśiṃ vada pṛcchataḥ ||
dvyādyaiḥ ṣalparyantairekāgraḥ ko bhavedrāśiḥ |
saptabhireva sa śuddho vada śīghraṃ gaṇaka rāśiṃ tam ||

Before we end this section, we would like to highlight an interesting and easy method for finding the square root and cube root that has been presented in the *Yogāśraya* commentary along with a lucid English rendering of the same [Ms 3585A ff. 28-29.]:

oru vargasaṃkhye mūlikkeṇam ennu varikil ā vargasaṃkhye veccu iṣṭamāyiṭṭu oru saṅkhye hārakamāyi kalpiccu mīte veccu harippū | haricca phalatte hārakattil kūṭṭi arddhippū | atine koṇṭu pinneyuṃ naṭette vargasaṃkhye tanne harippū | pinneyuṃ atukoṇṭu vargasaṃkhye harippū | ī vaṇṇaṃ hārakattinnu aviśeṣaṃ varuvoḷaṃ ceyyū | ennāl ā hārakaṃ tanne ā vargasaṃkhyeṭe mūlamennṟika |

pinne ghanatte mūlippānuṃ ghanasaṃkhye veccu ī vaṇṇaṃ vallatuṃ oru hārakatte kalpiccu harippū | hariccuṇṭāya phalatte pinneyum ā hārakaṃ koṇṭu tanne harippū | aviṭe uṇṭāya phalatte hārakattil kūṭṭi ardhippū | pinne atine hārakamāyi kalpiccu harippū aviśeṣaṃ varuvoḷaṃ | ennāl ghanamūlavuṃ varuṃ |

Divide the number [whose square root is to be found] by any chosen number. Then find the average of the divisor and quotient. [If the average is not a whole number, leave aside the remainder]. Repeat the process, treating this average as the new divisor till you do not get a remainder. At this stage the divisor and quotient will be the same, which is the square root.

In the case of a cube root, divide the given number by any chosen number. Then divide the quotient obtained again by the same chosen number [of course neglecting the remainder]. Then find the average of the divisor and quotient. [Leaving aside the remainder if any,] repeat the process, treating this average as the new divisor till you do not get a remainder. The quotient will be the cube root.

We now illustrate the procedure outlined above with examples, one for the square root and one for the cube root.

A. Suppose we want to find out the square root of 81.

Divide 81 by any chosen number, say, 3. We obtain the quotient 27. Now the average of divisor and quotient would be $\frac{(27+3)}{2}$, i.e. 15. Now dividing 81 by this new divisor, 15 we get the quotient 5 and a remainder 6. The average of this new quotient and the divisor is $\frac{(15+5)}{2}$, i.e., 10 [leaving aside the remainder] form the new divisor.

Now dividing 81 by this new divisor 10, we obtain 8 as the quotient and 1 as the remainder. The average of this divisor and the quotient $\frac{(10+8)}{2}$ i.e., 9 form the new divisor.

Dividing, again 81 by this new divisor, i.e. 9, we get the quotient 9 without any remainder. This last divisor (the same as the present quotient), 9 is the square root.

B. Now let us find out the cube root of 512.

Dividing this by a chosen number, say 3, we get 170 with the remainder 2. Dividing the quotient (leaving aside the remainder) 170 by the same divisor, i.e. 3, we get 56 with remainder 2. Again, leaving aside the remainder, find the average of the divisor and the quotient, which would be the new divisor $\frac{(3+56)}{2}$, i.e. 29 (neglecting the fractional part).

Now dividing the original number 512 by this new divisor 29, we get 17 with a remainder 19. As we cannot divide the quotient, 17, by the same divisor, 29 we take zero as the quotient and 17 as the remainder. Now the average of divisor and quotient obtained would be $\frac{(29+0)}{2}$, i.e. 14 (neglecting the fractional part).

Now dividing again the original number 512 by this new divisor, 14, we get 36 with a remainder of 8. Dividing the quotient (leaving aside the remainder) 36 again by 14, we get 2 as quotient with a remainder of 8. Now the average of the divisor and quotient would be $\frac{(14+2)}{2}$, i.e., 8.

Now dividing the original number 512 by this new divisor 8, we get 64. Again dividing the quotient by the same divisor 8, we get 8 as quotient with zero remainder. This quotient 8 is the same as the divisor. And it is the cube root of the given number 512.

4 Variant readings and their analysis

Both the commentaries give an abundance of variant readings. As it is impossible to give an exhaustive list, some important variants are given below.

1. In the *BV* and the *KKK* the characteristic feature (*lakṣaṇa*) of an impossible figure (*akṣetra*) are given as:

   धृष्टोद्दिष्टमृजुभुजं क्षेत्रं यत्रैकबाहुतः स्वल्पा ।
   तदितरभुजयतिरथवा तुल्या ज्ञेयं तदक्षेत्रम् ॥

   dhṛṣṭoddiṣṭamṛjubhujaṃ kṣetraṃ yatraikabāhutaḥ svalpā
   taditarabhujayatirathavā tulyā jñeyaṃ tadakṣetram ||

   स्वल्पा तदितरभुजयुतिरथवा तुल्यैकभुजमानात् ।
   उद्दिष्टा यदि मोहान्नेदृक् क्षेत्रं भवत्यतोऽक्षेत्रम् ॥

> *svalpā taditarabhujayutir athavā tulyaikabhujamānāt |*
> *uddiṣṭā yādi mohānnedṛk kṣetraṃ bhavatyato'kṣetram ||*
>
> If, by ignorance, a plane figure is so desired that one of its sides is either equal or greater than the sum of other sides (in measure), then it is a non-figure (*akṣetram*), for there cannot be such a figure.

The *KKK* gives another *lakṣaṇa* which runs as:

> चतुरश्रे त्र्यश्रे वा क्षेत्रे यत्रैकबाहुतः स्वल्पा ।
> तदितरभुजयुतिरथवा तुल्या ज्ञेयं तदक्षेत्रम् ॥
>
> *caturaśre tryaśre vā kṣetre yatraikabāhutaḥ svalpā |*
> *taditarabhujayutir athavā tulyā jñeyaṃ tad akṣetram ||*
>
> If, in a quadrilateral or a triangle, the sum of the other sides is either less or equal to any of its sides, then it should be known as a non-figure.

This is followed by the statement '*iti keṣāñcitpāṭhaḥ*' (this is the reading of some people). The *Yogāśraya* gives the second verse as the *lakṣaṇa* and does not give the first verse at all. The *Abhipreta* gives both the verses in the order of the *KKK*.

2. The *KKK*, after giving the formula to find out the diagonals of a trapezium as:

> कर्णाश्रितस्वल्पभुजैक्यमुर्वीं प्रकल्प्य तच्छेषभुजौ च बाहू ।
> साध्योऽवलम्बश्च तथान्यकर्णः स्वोर्व्याः कथञ्चिच्छ्रवणो न दीर्घः ॥
> तदन्यलम्बाच्च लघुस्तथेदं ज्ञात्वेष्टकर्णः सुधिया विभाव्यः ।
>
> *karṇāśritasvalpabhujaikyamurvīṃ prakalpya taccheṣabhujau ca bāhū*
> *sādhyo'valambaśca tathānyakarṇaḥ syorvyāḥ kathañcicchravaṇo na dīrghaḥ ||*
> *tadanyalambācca laghustathedaṃ jñātveṣṭakarṇaḥ sudhiyā vibhāvyaḥ |*

gives a variant reading, probably for the second hemistich of the first verse and the first part of the third hemistich, as:

> साध्योऽवलम्बो लघु दोस्समानादूनोंत्र कर्णो न समो न दीर्घः ।
> अन्यस्तु लम्बाधिकः ।
>
> *sādhyo'valambo laghu dossamānādūno'tra karṇo na samo na dīrghaḥ |*
> *anyastu lambādhikaḥ |*

with the remark '*iti keṣucit pustakeṣu pāṭho dṛśyate*' (some texts have this reading). The *Yogāśraya* gives this variant reading in the same manner, with a slight difference (*anyastu lambādadhikaḥ*). The only difference is that the variant reading is preceded with the remark '*atra pāṭhāntaram*'.

In the section on the right triangle, the *KKK* gives two alternate *karaṇasūtras* with the remark '*evaṃ vā karaṇasūtram*' [Līlā1975, pp. 309–10]. These verses do not find a place in either *Yogāśraya* or *Abhipreta*.

Vaṭaśśeri Parameśvara's formula mistaken as a *Līlāvatī* passage

We have seen that the *KKK*, *Yogāśraya* and *Abhipreta* have the *karaṇasūtras* viz. '*doṣṇāṃ*' etc. to find the diameter of a circle from the four sides of an inscribed quadrilateral, which is absent in the BV. These two verses are actually prescribed by Vaṭaśśeri Parameśvara in his commentary on the *Līlāvatī*. The fact that Parameśvara has given such a formula in his commentary was pointed out by T. A. Saraswati Amma in 1962.[11]

K. V. Sarma, in his work, *A History of the Kerala School of Hindu Astronomy* cites these verses with the prelude [Sar1972, p. 19]:

> In Western Mathematics the eighteenth century mathematician Lhuilier is credited with the discovery, in 1782, of an expression for the circum-radius of a cyclic quadrilateral. In India, however, we find the same formula enunciated by the Kerala astronomer Parameśvara (c. 1360-1455) in his commentary on *Līlāvati*, in the following lines:

This he wrote in the year 1972. But in his edition of the *KKK* (1975), he has given these very same verses as the 190th *karaṇasūtram* (as two numbers, 190a and 190b) of the text *Līlāvatī* [Līlā1975, p. 363], with the introductory statement '*atha catuṣkoṇaspṛśaḥ paridher vyāsārdhakalpanāya karaṇasūtraṃ*'.

In this case, Sarma has not noticed that these verses are the very same ones which he had stated earlier to be enunciated by Parameśvara. It is quite possible that having seen the introductory statement '*atha catuṣkoṇaspṛśaḥ paridher vyāsārdhakalpanāya karaṇasūtram*' in the manuscript, he might have taken these lines as belonging the *Līlāvatī*.

The very same verses appear in the *Yuktidīpikā* commentary (on the *Tantrasaṅgraha*) of Śaṅkara Vāriyar as well. There, it is included in the long tract of verses dealing with cyclic quadrilaterals, of course without any attribution or introduction. This commentary was also edited by K. V. Sarma in 1977.

In the mathematical tradition of medieval Kerala many findings that were couched in beautiful Sanskrit verses which seem to have been a part of the oral tradition (for instance verses presenting infinite series) are not found to be part of any of their own works that are extant today. We know this fact

[11] She mentions this in the article 'The development of mathematical ideas in India' [Sar1962], (quoted by K. V. Sarma, see [Sar1972, p. 20]). She repeats this in her work *Geometry in Ancient and Medieval India*, see [Sar1999, p. 108].

from the path-breaking studies of Sarma himself. Many such verses containing mathematical findings of *Saṅgamagrāma Mādhava* are preserved in the works of Vaṭaśśeri Parameśvara, Keḷallūr Nīlakaṇṭha Somayājī, Jyeṣṭhadeva, Putumana Somayājī and others.

In the *KKK* itself, we get many such verses prescribing mathematical formulae, which are introduced as *karaṇasūtras* or *sūtras*. Such verses gained much popularity in the oral tradition and in the course of time the original promulgator was forgotten. As a result, some of these were mistaken as part of some classical texts.

It may further be noted in this connection that the *Yogāśraya* commentary seems to give yet another *karaṇasūtra* at the end of the exposition on '*doṣṇāṃ*' etc. The passage runs as: '*tryaśre karaṇasūtram śūnya(ṃ?) prakalpya turyandostryaśre vetat*[12] *samaṃ bhavet'*. Interestingly Vaṭaśśeri Parameśvara's commentary on the *Līlāvatī* has the same passage.[13]

Other verses of Parameśvara

We have noticed that the *Abhipreta* commentary has many common verses with the *KKK*. Actually many of these verses are by/of Vaṭaśśeri Parameśvara. This has been mentioned by Nārāyaṇa (of Mahiṣamaṅgalam), the joint author of the *KKK*. Thus while commenting on the 233rd verse (*chāyāvyavahāra*) of *Līlāvatī*, he writes [Līlā1975, p. 424]:

> अत्र द्वादशाङ्गुलशङ्कोः कालद्वयसम्भूतच्छायायोगतदन्तरकर्णयोगतदन्तरेषु द्वाभ्यां ताभ्याम् अज्ञातयोरितरयोरानयनं षोढा कार्यम् इत्युक्तं युक्तिमार्गेण, इत्यस्यार्थं विवृण्वता परमेश्वराचार्येण। तेनैव करणसूत्राणि च कृतानि। तानि सोदाहरणानि सन्यासानि च मया प्रदर्श्यन्ते।
>
> *atra dvādaśāṅgulaśaṅkoḥ kāladvayasambhūtacchāyāyogatadantarakarṇayogatadantareṣu dvābhyāṃ tābhyām ajñātayor itarayor ānayanaṃ ṣoḍhā kāryam ityuktaṃ yuktimārgeṇa, ityasyārthaṃ vivṛṇvatā parameśvarācāryaṇa | tenaiva karaṇasūtrāṇi ca kṛtāni | tāni sodāharaṇāni sanyāsāni ca mayā pradarśyante |*
>
> Parameśvara has said, while interpreting the word '*yuktimārgeṇa*' (in *Līlāvatī* 233), that in six ways one should find out the third quantity from the given two, of the triad – 1) the sum of shadows of the gnomon of 12 *aṅgulas*, at two points of time, 2) the sum of hypotenuses formed by the shadows, and 3) the difference between shadows. He has composed *karaṇasūtras* as well. These are shown [hereunder] by me together with illustrative examples and [their solutions beginning with] *nyāsas* (setting-downs).

Following this, he gives six formulae and their illustrative examples. Thirteen verses of this tract are reproduced in the *Abhipreta*.

[12] It seems that there is a scribal error. Ms No. 3585A has this reading clearly.

[13] See Ms No. 498B of Thunchan Manuscripts Repository, University of Calicut, folio no. 103.

Verses of an unknown scholar

Similarly at the end of the section on '*chāyādyānayana*', Nārāyaṇa quotes the formulae and illustrative examples of a scholar with the introductory remark [Līlā1975, p. 435]:

> अथ यत्र दीपस्तम्भे द्रढीकृतस्य गोलस्य छाया समवृत्ता तत्र गोळमध्यादूर्ध्वदीपमस्तकान्तस्य प्रदेशस्य, परिमाणं, दीपस्तम्भपरिमाणं, गोळव्यासः, छायाव्यास इत्येवं चतुर्षु सत्सु तेषु विदितैस्त्रिभिः अज्ञातस्येतरानयनं चतुर्धा कार्यम् इत्युपदिष्टं केनचिद् गणितयुक्तिविदग्रेसरेण। तदप्यत्र दर्श्यते।
>
> *atha yatra dīpastambhe draḍhīkṛtasya golasya chāyā samavṛttā tatra goḷamadhyādūrdhvadīpamastakāntasya pradeśasya, parimāṇām, dīpastambhaparimāṇam, goḷavyāsaḥ, chāyāvyāsa ityevaṃ caturṣu satsu teṣu viditais tribhiḥ ajñātasyetarānayanaṃ caturdhā kāryam ityupadiṣṭaṃ kenacid gaṇitayuktividagresareṇa | tadapyatra darśyate |*
>
> A scholar, the foremost of those who know the rationales of mathematics has taught that the calculation of the fourth quantity should be made in four ways from the given three of the tetrad –
>
> 1. the extension of the lamp post above the fixed sphere,
> 2. the total length of the lamp post,
> 3. the diameter of the sphere,
> 4. the diameter of the shadow,
>
> where the shadow of a sphere fixed on a lamp post is uniformly circular. These methods are shown here.

He then gives four formulae and three illustrative examples. These verses are also reproduced in the *Abhipreta* commentary. Further investigations are to be carried out to get the identity of the scholar. These *sūtras* and *udāharaṇas* which might have been very popular in the oral tradition of the Kerala School of Mathematics clearly indicate ample scope for further research in the field.

The common verses (outside the text of the *Līlāvatī*) that are seen in the *KKK* and the *Abhipreta* commentary have, in fact, this oral tradition as the source. Even the verses which are ascribed to Parameśvara by tradition may be due to an earlier author belonging to this tradition and might not have actually been composed by him.

Correct reading is preserved

At the end portion of the section on *kuṭṭaka*, the *KKK* gives two examples for finding:

1. the number of *adhimāsas* and *sauramāsas* that have elapsed from the remaining *adhimāsas* and
2. the number of *avamas* (omitted lunar days) and lunar months that have elapsed from the *avamaśeṣa* (the remaining *avamas*).

K. V. Sarma has edited the first verse filling the lacunae in the manuscripts as [Līlā1975, p. 453]:

(ख्याता यौगाधिमा) सा नववसुहुतभुक् षड्रसैः सम्मिताः स्युः
सौरा मासास्तदीया अयुतहतरसैकाश्विनो यत्र विद्वन् ।
शेषो यत्राधिमासस्य तु खनगगजस्सप्तशून्यद्विसंख्या
(स्तस्माद् याताधिमासान् गणक) वद तथा सौरमासांश्च (यातान्) ॥

(khyātā yaugādhimā) sā navavasuhutabhuk ṣaḍrasaiḥ sammitāḥ syuḥ
saurā māsāstadīyā ayutahatarasaikāśvino yatra vidvan |
śeṣo yatrādhimāsasya tu khanagagajassaptaśūnyadvisaṃkhyā-
(stasmād yātādhimāsān gaṇaka) vada tathā sauramāsāṃśca (yātān) ||

We get the correct reading of the complete verse from the *Abhipreta* commentary as:

यौगा यत्राधिमासा नव वसुहुतभुक् षड्रसैः सम्मिताः स्युः
सौरा मासास्तदीया अयुतहतरसैकाश्विनो यत्र विद्वन् ।
शेषो यत्राधिमासस्य खनगगजसप्ताङ्गशून्यद्विसङ्ख्याः
तत्रातीताधिमासान् गणक वद तथा सौरमासांश्च यातान् ॥

yaugā yatrādhimāsā nava vasuhutabhuk ṣaḍrasaiḥ sammitāḥ syuḥ
saurā māsāstadīyā ayutahatarasaikāśvino yatra vidvan |
śeṣo yatrādhimāsasya khanagagajasaptāṅgaśūnyadvisaṅkhyāḥ
tatrātītādhimāsān gaṇaka vada tathā sauramāsāṃśca yātān ||

In the second verse also Sarma faces difficulties without getting the correct reading. He notes that the number as given in the explanatory portion does not agree with that which is given in the verse. His version of the verse is as follows [Līlā1975, p. 454]:

यौगानीहावमानि त्रियुगखगजैकाब्धिसङ्ख्यानि विद्वन्
चान्द्राहाङ्काङ्गाङ्गा रसविधुनगाङ्गाश्वितुल्यानि यत्र ।
अष्टाभ्राह्येकदस्रोदधिशरयुगळआश्चावमस्यात्र शेषः
तत्रातीतावमानि प्रवद मम गतानीह चान्द्रान्यहानि ॥

yaugānīhāvamāni triyugakhagajaikābdhisaṅkhyāni vidvan
cāndrāhāṅkāṅgāṇgā rasavidhunagāṅgāśvitulyāni yatra |
aṣṭābhrāhyekadasrodadhiśarayugaḷāścāvamasyātra śeṣaḥ
tatrātītāvamāni pravada mama gatānīha cāndrānyahāni ||

Here the lunar days in a *Yuga* are given by the phrase '*aṅkāṅgāṅgā rasavidhunagāṅgāśvitulyāni*'. The number, as per the reading, is 2, 67, 16, 669. Sarma

notes that in the manuscript this is wrongly decoded as 2, 67, 16, 668. But, in fact, problem is with Sarma's reading of the verse. The number given in the manuscript is right. The *Abhipreta* commentary has the correct reading of the second line as '*cāndrāhānyaṣṭakāṅgāṅga rasavidhunagāṅgāśvitulyāni yatra*', which solves the problem.

5 Conclusion

Thus we can see that the two Malayalam commentaries provide valuable source material for further research on the textual tradition of the *Līlāvatī*. These also affirm the scope of further research on the oral tradition of medieval Kerala mathematics.

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Mensuration of quadrilaterals in the *Līlāvatī*

S. G. Dani*

Mensuration with quadrilaterals had received attention in the *siddhānta* tradition at least since Brahmagupta. However, in Bhāskarācārya's *Līlāvatī* we come across some distinctively new features. In this paper an attempt will be made to put the development in historical perspective.

A systematic study of the topic of mensuration of quadrilaterals in Indian mathematics goes back at least to the *Brāhmasphuṭasiddhānta* (628 CE) of Brahmagupta (born in 598 CE); in certain special cases, such as isosceles trapezia, some familiarity is found in the *Śulvasūtras* from around the middle of the first millennium BCE [SB1983]. With regard to mensuration of quadrilaterals Brahmagupta is well-known for the formula given in the *sūtra*:

स्थूलफलं त्रिचतुर्भुजबाहुप्रतिबाहुयोगदलघातः ।
भुजयोगार्धचतुष्टयभुजोनघातात् पदं सूक्ष्मम् ॥

sthūlaphalaṃ tricaturbhujabāhupratibāhuyogadalaghātaḥ |
bhujayogārdhacaṭuṣṭayabhujonaghātāt padaṃ sūkṣmaṃ ||

[BSS1902, ch. 12.21]

Traditionally the *sūtra* has been understood, by ancient mathematicians following Brahmagupta (I shall dwell more on this later), as well as modern commentators broadly as follows:

> The gross area of a triangle or quadrilateral is the product of half the sum of the opposite sides. The exact area is the square-root of the product of the four sets of half the sum of the sides (respectively) diminished by the sides.

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This translation is taken from [Sar1999, p. 88], and the translations in [Col1817] and [Plo2009] also correspond to it. In all of these, in particular, *tricaturbhuja* is interpreted as referring to the formula being applicable to triangles and quadrilaterals (independently). For the case of the triangle this is the well-known formula, known after Heron of Alexandria (first century CE), and in this context the quadrilateral version is referred to as "Brahmagupta's generalization" (see, for instance, [Plo2009, p. 144]). The general version, which in modern notation may be stated as

$$A = \sqrt{(s-a)(s-b)(s-c)(s-d)}, \qquad (*)$$

where a, b, c, d are the sides of the quadrilateral, s is half the perimeter, and A is the area, is however correct only when the quadrilateral is cyclic, viz. when all the four vertices lie on a common circle; this condition holds for quadrilaterals like rectangles and isosceles trapezia, but not in general, e.g. for rhombuses with unequal diagonals. The general perception in the context of the interpretation has been that the author somehow omitted to mention the condition, though actually aware of it, with the latter being confirmed by the fact that he is noted to apply it only for cyclic quadrilaterals.

It has however recently been argued in [Kic2010] that actually the term *tricaturbhuja* was in fact used by Brahmagupta to mean a cyclic quadrilateral (and *not* triangle and/or quadrilateral). Thus Brahmagupta means to state:

> The gross area of a **cyclic quadrilateral** is the product of half the sum of the opposite sides, and the square-root of the product of the four sets of half the sum of the sides (respectively) diminished by the sides is the exact area.

A hint that the traditional interpretation may not be right is contained in the fact that at the only other place where the term *tricaturbhuja* occurs in *Brāhmasphuṭasiddhānta* [BSS1902, ch. 12.27], (and it is not known to occur in earlier or later ancient texts), the result involved (relating to the circumradius) is stated first for triangles, separately, and then for "*tricaturbhuja*"s which indicates that the latter should in fact have four sides [Kic2011]. The arguments in Kichenassamy go well beyond that, with the author providing a detailed discussion on the issue, including on how Brahmagupta would have arrived at the formula, and how it incorporates in a natural way the hypothesis that the quadrilateral is cyclic. According to Kichenassamy, Brahmagupta while pursuing his study of triangles dealt with the circumcircle, described in particular a formula for the circumradius, and along this line of thought considered quadrilaterals formed by the triangle and a point on the circumcircle, which motivated the term *tricaturbhuja*.[1]

[1] Recently P. P. Divakaran [Div2018] has proposed (see Chapter 8, Sections 4 and 5; my thanks to Divakaran for the pre-publication information) a somewhat different scenario

Unfortunately, the theory developed by Brahmagupta did not go down the line of later mathematicians in India with proper understanding. It may be worthwhile to recall the following in this respect. Let us consider the works of Śrīdhara, the author of *Pāṭīgaṇita* [PāGa1959] and *Triśatikā*, and Mahāvīra who authored *Gaṇitasārasaṅgraha* [GSS2000], two prominent authors[2] from the intervening period between Brahmagupta and Bhāskara. Mahāvīra is known to be from around 850 CE. Concerning Śrīdhara there has been a controversy among scholars over his period, and in particular over whether he was anterior or posterior to Mahāvīra, but it now seems to be agreed that he is from the eighth century.[3]

In the *Pāṭīgaṇita*, on the issue of areas of quadrilaterals, Śrīdhara first gives a formula for the areas of trapezia (the usual one), in *sūtra* 115 which is then complemented, in *sūtra* 117 [PāGa1959, p. 175], with the following:

> भुजयुतिदलं चतुर्धा भुजहीनं तद्वधात् पदं गणितम् ।
> सदृशासमलम्बानां असदृशलम्बे विषमबाहौ ॥
>
> *bhujayutidalaṃ caturdhā bhujahīnaṃ tadvadhāt padaṃ gaṇitam* |
> *sadṛśāsamalambānāṃ asdṛśalambe viṣamabāhau* ||
>
> [PāGa1959, v. 117]
>
> Set down half the sum of the [four] sides [of the quadrilateral] in four places, [then] diminish them [respectively] by the [four] sides [of the quadralateral], [then] multiply [the resulting numbers] and take the square root [of the product]: this

for the development of ideas in Brahmagupta's work and the genesis of the term *tricaturbhuja*, which nevertheless discounts the traditional interpretation of the term mentioned earlier.

[2] As noted by K. S. Shukla in the introduction to his edition of Śrīdhara's *Pāṭīgaṇita*, Śrīdhara's works are cited by many later authors. On the other hand, Mahāvīra's *Gaṇitasārasaṅgraha* apparently enjoyed the status of a textbook in many parts of South India for nearly three centuries, until the arrival of Bhāskarācārya's *Līlāvatī*, as noted by Balachandra Rao, in his review [Rao2013, p. 167] of the book *Śrī Rājāditya's Vyavahāragaṇita* edited and translated by Padmavathamma, Krishnaveni and K. G. Prakash.

[3] In §5 of Shukla's introduction to his edition of Śrīdhara's *Pāṭīgaṇita*, one finds a detailed discussion on this issue, concluding with his own verdict that Śrīdhara "lived sometime between Mahāvīra (850) and Āryabhaṭa II (950)". Saraswati Amma expresses skepticism in this respect [Sar1999, p. 10]; her wording is "Śrīdhara is probably earlier than Mahāvīra though K. S. Shukla places him between 850 and 950 AD". S. D. Pathak, in his paper *Śrīdhara's time and works*, [Pat2003], which was published posthumously but was actually written before Shukla's edition of *Pāṭīgaṇita* was published, argues in favour of Śrīdhara being earlier, and, notably, in a special note following the article, the editor R. C. Gupta, who had himself also discussed the issue in an earlier paper, mentions "But now Dr. Shukla himself accepts the priority of Śrīdhara over Mahāvīra (personal discussions)". Also, Shefali Jain in her recent thesis [Jai2013] mentions that at the end of a manuscript of the *Gaṇitasārasaṅgraha* in the Royal Asiatic Society, Bombay [Ms-GSS] one finds the statement "*kramādityuktaṃ śrīdharācaryeṇa bhadraṃ bhūyāt*" which shows that Śrīdhara preceded Mahāvīra. In the light of earlier literature she assigns 750 CE as the year around which he would have flourished.

> gives the area of the quadrilaterals having [two or more] equal sides but unequal altitudes and also of quadrilaterals having unequal sides and unequal altitudes.
>
> (Translation from [PāGa1959]).

Thus Śrīdhara is seen to give formula (∗) for the area of any quadrilateral, dwelling elaborately on the generality, without realising that it is not true in that generality. Interestingly, unlike in the case of the formula for trapezia, no examples are discussed to illustrate the general formula. This suggests in a way that the *sūtra* is included in the spirit of recording and passing on a piece of traditional knowledge in which the author espouses no direct interest; this is a kind of situation in which mathematicians are prone to let down their guard! His treatment in the *Triśatikā* is also along the same lines [Sar1999, p. 92].

Mahāvīra states the result under discussion as follows:

> भुजायुत्यर्धचतुष्कात् भुजहीनाद्घातितात्पदं सूक्ष्मम् ।
>
> *bhujāyutyardhacatuṣkāt bhujahīnād ghātitāt padam sūkṣmaṃ* |
>
> [GSS2000, ch. 7.50]
>
> Four quantities represented [respectively] by half the sum of the sides as diminished by [each of] the sides [taken in order] are multiplied together and the square root [of the product so obtained] gives the minutely accurate measure [of the area of the figure]. (Translation from [GSS2000])

Here again, formula (∗) is stated unconditionally for any quadrilateral. The second half of the above mentioned verse is the usual formula for the area of a trapezium, as the product of the perpendicular height with half the sum of the base and the opposite side, mentioning also a caveat that it does not hold for a *viṣamacaturasra* (a general quadrilateral with unequal sides). In verses 51–53 following the *sūtra* as above, Mahāvīra asks to compute the areas of triangles with given lengths for their sides, presumably meant to be done using the first part of verse 50. Verse 54 describes a formula for the diagonal of a quadrilateral[4] and in verses 55 to 57 the author asks to compute diagonals and areas of quadrilaterals which are isosceles trapezia; since reference to diagonals is also invoked it is not clear whether the computation is meant to be done using the general form of verse 50, namely formula (∗), or by first computing the diagonal. As a whole the treatment suggests either lack of interest or disbelief in the general case of the formula.

Similarly, Śrīpati (eleventh century) also gives, in the *Siddhāntaśekhara*, formula (∗) unconditionally [Sar1999, p. 94]. On the whole the practice of treating the expression as the formula for the area of any quadrilateral was so

[4] This formula also goes back to Brahmagupta and is valid only for cyclic quadrilaterals, but is stated here unconditionally.

prevalent that one finds it presented as such even in the fourteenth century work *Gaṇitasārakaumudī* of Ṭhakkura Pherū [GSK2009, p. 142]. Notwithstanding the overall continuity of the Indian mathematical tradition, topics that were not directly involved in practice, in astronomy or other spheres in which mathematics was applied at the time, suffered neglect, and sometimes were carried forward without a proper understanding of what was involved.

By the time of Bhāskara any connection of the formula $(*)$ with cyclicity of the quadrilateral was completely lost. Even awareness of cyclic quadrilaterals seems to have gone missing over a period, until it was resurrected in the work of Nārāyaṇa Paṇḍita, in the fourteenth century [Sar1999, pp. 96–106].

In this overall context, as it prevailed around the turn of the millennium, Āryabhaṭa II, who is believed to have lived sometime between 950 and 1100 CE [Plo2009, p. 322] rejected $(*)$ as the formula for the area of a quadrilateral, ridiculing one who wants to find the area of a quadrilateral without knowing the length of a diagonal as a fool or devil (*mūrkhaḥ piśāco vā*) [Sar1999, p. 87]. This was the situation when Bhāskara appeared on the scene. Though apparently guided by Āryabhaṭa II in his treatment in respect of Brahmagupta's theorem [Sar1999, p. 94], Bhāskara took an entirely different approach to the issue, bringing considerable clarity on the topic (even though he did not get to cyclic quadrilaterals).

A closer look at the relevant portion of the *Līlāvatī* shows an intense concern on the part of Bhāskara at what he observed as a flaw in the "traditional" formula. This does not seem to have been adequately appreciated in the literature on the topic. One of the reasons for this seems to be that the standard translation and commentaries (E.g. [Col1993], [Pha2014] [Līlā2001], and [Līlā2008]) which have been the chief sources for dissemination of the topic, have translated and commented upon the *sūtras* involved only individually, in a rather disjointed way, as a result of which a common strand that Bhāskara followed in respect of the above seems to have been missed. Secondly, many of the commentaries, except [Col1993] from the above, while including the text of the *Līlāvatī*, do not include Bhāskara's *Vāsanābhāṣya* (explanatory annotation, in prose form) along with it, and even as they are seen to avail of various points made in *Vāsanābhāṣya*, no reference is made to the latter, which diffuses the overall picture even further. Colebrooke does include the *Vāsanābhāṣya* and also a meticulous translation of it for the most part, though as I shall point out below, a crucial line relevant to the theme under discussion is missing from the translation.[5]

I shall now present the part of the *Līlāvatī* together with the *Vāsanābhāṣya* on the issue as above and bring out the strand of Bhāskara's thinking, and

[5] Similarly, some other short segments aimed at introducing the subsequent verse have been omitted from the translation elsewhere in the text.

concern, over the perceived flaw. Let me begin with Bhāskarācārya's statement on the formula.[6]

सर्वदोर्युतिदलं चतुःस्थितं बाहुभिर्विरहितं च तद्वधात् ।
मूलमस्फुटफलं चतुर्भुजे स्पष्टमेवमुदितं त्रिबाहुके ॥

sarvadoryutidalaṃ catuḥsthitaṃ bāhubhirvirahitaṃ ca tadvadhāt |
mūlamasphuṭaphalaṃ caturbhuje spaṣṭamevamuditaṃ tribāhuke ||

[Līlā1937, v. 169]

Half the sum of all the sides is set down in four places; and the sides are severally subtracted. The remainders being multiplied together, the square root of the product is the area, inexact in the quadrilateral, but pronounced exact in the triangle.[7]

Thus, unlike Āryabhaṭa II, Bhāskara does not reject outright the formula for the quadrilateral, accepting it wholly only for triangles. He mentions it as exact (*spaṣṭa*) for triangles while for quadrilaterals he calls it "inexact" (*asphuṭa*). This is however only the beginning. Detailed comments on it are to follow.

To begin with he asks, in verse 170, for the area to be computed "as told by the ancients" (*tatkathitaṃ yadādyaiḥ*) for a quadrilateral with base (*bhūmi*) 14, face (*mukha*) 9, sides 13 and 12, and perpendicular 12; it can be seen that the quadrilateral is a non-isosceles trapezium. In the *Vāsanābhāṣya* he proceeds to note that the area given by the formula is $\sqrt{19800}$, which is "a little less than 141", while the actual area (which can be computed for a trapezium more directly) is 138, thus pointing to a contradiction.

Though this would have sufficed for the contention that the formula is not correct, or accurate, Bhāskara does not leave it at that. He embarks on a more detailed discussion on the issue. At the beginning of verse 171, in the *Vāsanābhāṣya* he says:

अथ स्थूलत्वनिरूपणार्थं सूत्रं सार्धवृत्तम् ।

atha sthūlatvanirūpaṇārthaṃ sūtraṃ sārdhavṛttaṃ |

Now a *sūtra* of a stanza and half for looking into the grossness.

[6] For reference to the verses I shall indicate their numbers as in [Līlā1937], the standard Sanskrit sourcebook on the topic; the reader is cautioned that the numbers in different references vary somewhat from each other. In the *sūtras* involved here the differences are small.

[7] The translation is taken from [Col1993]; I have added a comma, after "area", which seems to be needed for easy comprehension.

This line, which is significant from our point of view, is not found in the translation in [Col1993] (as noted earlier many other sources do not include the text of the *Vāsanābhāṣya* either). We note that "*atha*" marks the commencement (of a story, chapter, argument etc., typically in a ceremonial way) and that *nirūpaṇa* means "looking into, analysis or investigation". *Sthūlatva* stands for "grossness" or "coarseness"; thus Bhāskara is announcing here that he is taking up an analysis of the grossness[8] (of the formula). This is followed by the following argument [Līlā1937, v. 171]:

चतुर्भुजस्यानियतौ हि कर्णौ कथं ततोऽस्मिन्नियतं फलं स्यात् ।
प्रसाधितौ तच्छ्रवणौ यदाद्यैः स्वकल्पितौ तावितरत्र न स्तः ॥
तेष्वेव बाहुष्वपरौ च कर्णौ अनेकधा क्षेत्रफलं ततश्च ॥

caturbhujasyāniyatau hi karṇau kathaṃ tato'sminniyataṃ phalaṃ syāt |
prasādhitau tacchravaṇau yadādyaiḥ svakalpitau tāvitaratra na staḥ ||
teṣveva bāhuṣvaparau ca karṇāu anekadhā kṣetraphalaṃ tataśca ||

The diagonals of a quadrilateral are indeterminate (*aniyatau*); then how could the area [confined] within be determinate? The (values for) diagonals set down (*prasādhitau*) by the ancients[9] would not be the same elsewhere. For same (choices of the) sides, the diagonals have many possibilities and the area would vary accordingly.

In the *Vāsanābhāṣya* this is further elaborated, noting that if in a quadrilateral two opposite vertices are moved towards each other then the diagonal between them contracts, while the other two vertices move away from each other and the diagonal between them elongates, and thus with sides of the same length there are other possible values for the diagonals.

The next verse in the *Līlāvatī* [Līlā1937, v. 173] raises some rhetorical questions:

लम्बयोः कर्णयोर्वैकं अनिर्दिश्यरपरः कथम् ।
पृच्छत्यनियत्वेऽपि नियतं चापि तत्फलम् ॥

lambayoḥ karṇayorvaikaṃ anirdiśya paraḥ kathaṃ |
pṛcchatyaniyatatvepi niyataṃ cāpi tatphalaṃ ||

When none of the perpendiculars nor either of the diagonals are specified, how will the other values get determined? It is like asking for definite area, when in fact it is indeterminate.

Then in verse 174 come some devastating blows:

[8] Like the English word adopted here for *sthūlatva*, the latter also has shades of meaning of unflattering variety, and one may wonder whether the choice of the word *sthūlatva* here, as against say the noun form of the adjective *asphuṭa* that was adopted in the original *sūtra*, is deliberate.

[9] The ancients refers to "Śrīdhara and the rest" according to Gaṇeśa's commentary; this has been noted in Colebrooke [Col1993, p. 112].

स पृच्छकः पिशाचो वा वक्ता वा नितरां ततः ।
यो न वेत्ति चतुर्बाहुक्षेत्रस्यानियतां स्थितिम् ॥

sa pṛcchakaḥ piśāco vā vaktā vā nitarāṃ tataḥ |
yo na vetti caturbāhukṣetrasyāniyatāṃ sthitiṃ ||

Such a questioner is a blundering devil (*piśāca*) and worse is one who answers it. They do not realise the indeterminate nature of the area of a quadrilateral.

The scorn being deployed is reminiscent of Āryabhaṭa II, but here we find it accentuated, and its scope expanded to those answering the question!

Having vented his ire over the ignoramuses, Bhāskara next sets out to establish the point through more concrete illustrations. Before continuing with it, it may be worthwhile to note the following. While the argument given in verse 171 is of considerable heuristic value, it is *not* conclusive from a logical point of view, since so far it has not been shown that when the sides are the same and diagonals vary, the areas could actually be different. To make the argument foolproof, and to fully convince skeptics, one needs concrete examples, with the same four sides and different pairs of diagonals for which the areas *are different*. While he may or may not have specifically followed such a train of thought, that is what Bhāskara sets out to do in the following verses.

For the illustrations he needs situations where after making alterations as proposed in the *Vāsanābhāṣya* following *sūtra* 171 it would be possible to readily compute the area (and show that it is different). For this purpose he considers the class of equilateral quadrilaterals, in which all sides are equal, also called rhombuses. He notes a formula for the second diagonal, given the common value of the side and one of the diagonals: in modern notation, if a is the side and d_1 and d_2 are the diagonals then

$$d_2 = \sqrt{4a^2 - d_1^2}.$$

He recalls also the formula for the area, as equal to $\frac{1}{2}d_1d_2$ in the verses that follow. The *sūtras* prescribing this also include a statement of the areas of rectangles and trapezia, but that is purely to put the situation of equilaterals in context - after all, the knowledge of these is already implicit in particular in the problem posed in verse 170.

At this juncture, one is in a position to illustrate the point that was made following verse 171, since if we start with an equilateral quadrilateral with side a and a diagonal of size d_1 and move the vertices on the two sides of that diagonal closer along the diagonal, then the area is $\frac{1}{2}d_1\sqrt{4a^2 - d_1^2}$, with d_1 as a variable quantity; one can readily see that choosing different values of d_1 we get rhombuses with the same side-lengths but different areas. However, not

content with this, he seeks more concrete choices for the side and diagonal, specifically with integer values.

For this purpose he brings in his knowledge of what are now called Pythagorean triples. A Pythagorean triple is a triple of natural numbers (l, m, n) such that $l^2 + m^2 = n^2$; by the Pythagoras theorem for a triangle with sides l, m and n where (l, m, n) is a Pythagorean triple, the angle opposite to the side of length n is a right angle. Putting four such right angled triangles together, along their equal sides adjacent to the right angles, we get an equilateral quadrilateral with all sides n and diagonals $2l$ and $2m$ respectively, and their areas are $2lm$. Bhāskara now chooses the triples $(15, 20, 25)$ and $(7, 24, 25)$ which are seen to be Pythagorean,[10] and have common value for the length of the hypotenuse. Thus the construction as above yields two equilateral quadrilaterals with areas $2(15 \times 20) = 600$ and $2(7 \times 24) = 336$, respectively. Bhāskara also points out that we may also consider for comparison the square with all sides equal to 25, in which case the area is 625, yet another value for the area. Thus Bhāskara adopts various means, argumentation, pressurising through rhetoric, as well as persuasion, to put it across to his readers that Brahmagupta's formula is not valid exactly for a general quadrilateral.

Following the group of verses discussed above, there are two more problems concerning the area of a quadrilateral. In verse 175 we have an example of (what turns out to be) a non-isosceles trapezium, for which again it is pointed out, in *Vāsanābhāṣya*, that the area computed using $(*)$ does not give the true value.

The next verse, 176, asks to find the area, and also the diagonals and perpendiculars, of a quadrilateral whose sides are given as, face 51, base 75, left side 68 and right side 40. To a modern reader it should seem puzzling that the Ācārya should ask such a question, giving only the sizes of the four sides, after all the painstaking endeavour to get it across that the sizes of the four sides do not determine a quadrilateral, and in particular the area is indeterminate. It is hard to reconcile this especially with verse 172 according to which one asking such a question is a "*piśāca*".

The spirit of what follows however seems to be to explain how one should proceed in response, when such a problem is posed (e.g. as a challenge); since at one time the focus was on cyclic quadrilaterals which were determined once the sides were given (together with their order), it may have been a general practice to pose questions about quadrilaterals purely in terms of their sides (in specific order). In the next verse Bhāskara recalls that if we know

[10] It may be recalled here as an aside that Pythagorean triples have been known in India since the time of the *Śulvasūtras*; for a discussion on such triples occurring in *Śulvasūtras* the reader is referred to [Dan2003].

the perpendicular, that determines the (corresponding) diagonal, and knowing a diagonal determines the (corresponding) perpendicular, and the area; this is consistent with the earlier contention about the need for an additional assumption being necessary. In his treatment of the problem in the *Vāsanābhāṣya* he then says "to determine the perpendicular we *assume* the diagonal joining the tip of the left side to the base of the right side to be (of length) 77" (emphasis added). With this choice for the diagonal, the perpendicular and then the area of the quadrilateral are computed (adding the areas of the two triangles on the two sides of the diagonal as above);[11] it turns out to be 3234. Interestingly, this is the value that one would get from $(*)$ with the values 51, 68, 75 and 40 for the four sides! One may wonder whether Bhāskara intended this, but there is no way to know. The reason for this agreement, from a modern perspective, is of course that for the above choice of the diagonal the quadrilateral is cyclic, and Brahmagupta's formula does apply.

It would seem curious that the choice made was such that the quadrilateral is cyclic, especially when there is no reference to such a concept in the text. Also, though the fact of having to make a choice has been clarified, one would wonder how the choice of 77 as the length of the (particular) diagonal came about, especially in the context of its turning out to be one for which the quadrilateral is cyclic. It may be noted that if one starts with an ad hoc choice, the computations of the perpendicular and the area involve rather complicated surds, making it unsuitable for an illustrative example. All commentators have repeated the part about assuming the (particular) diagonal to be 77 (generally without reference to the *Vāsanābhāṣya*), but throw no light on the issue of how the specific value may have come to be chosen. It seems that this quadrilateral was familiar to Bhāskara, together with the value for the diagonal. Brahmagupta had given a construction [BSS1902, ch. 12.38] of quadrilaterals with integer values for the sides and area, starting with a pair of Pythagorean triples (the reader is referred to [Pra2012] for an exposition on this), and it has been recalled in the *Līlāvatī* [Līlā1937, v. 189]. For each quadrilateral constructed using the Brahmagupta construction (which necessarily turns out to be cyclic) one gets some new ones (with vertices on the same circle as the original one) by replacing the triangle on one side of a diagonal by its reflection in the perpendicular bisector of the diagonal. This process of obtaining new quadrilaterals also turns up in the *Līlāvatī* (though there is no reference to their cyclicity along with it). Commentator Gaṇeśa has pointed out that the quadrilateral with sides 51, 68, 75 and 40 as in the

[11] Having assumed the diagonal to have length 77, the areas of the two triangles formed could be computed using $(*)$, which Bhāskara does consider to be exact for triangles, but the method given goes through the computation of the perpendicular, and no reference is made to this other possibility.

above discussion is one of the quadrilaterals arising in this way, starting with the Pythagorean triples $(3, 4, 5)$ and $(8, 15, 17)$ [Col1993, pp. 127–128].

Concluding remarks

To sum up, Bhāskara's response to the longstanding misunderstanding makes an interesting episode in the history of mathematics. While unfortunately he missed the fact that formula does hold if one restricts to cyclic quadrilaterals, and one may wonder if it was due to lack of access to the original formulation or due to lack of clarity in Brahmagupta's original version, his thought process and arguments would have provided a better understanding in the subsequent period, at least in some quarters though not universally (as seen earlier), on mensuration of quadrilaterals.

Aṅkapāśa in the *Līlāvatī*

Takanori Kusuba*

1 Introduction

While discussing the mixture procedure (*miśra-vyavahara*) in his *Līlāvatī*, Bhāskara gives the rule for the number of combinations of r things taken at a time from n things.

$$ {}_nC_r = \frac{n}{1} \cdot \frac{n-1}{2} \cdot \frac{n-2}{3} \cdots \frac{n-(r-1)}{r}. $$

He applies this rule to prosody (metrics), architecture (windows) and medicine (taste). His predecessors in *jyotiṣa* give this rule.[1]

At the end of the *Līlāvatī*, Bhāskara adds a new subject called *aṅkapāśa*. He concatenates digits and treats them as a number. He presents four rules for permutation with examples, which do not appear in any of his predecessors' mathematical works. In what follows I study these rules and examples.

2 Bhāskara's rules for permutations

In this study I use two editions of the *Līlāvatī*, both of which are accompanied by commentaries. One edition is with the *Buddhivilāsinī* (1545) by Gaṇeśa Daivajña and a *vivaraṇa* (1587) by Mahīdhara [Līlā1937]. This also includes the auto-commentary by Bhāskara. The other is with the *Kriyākramakarī* (ca.

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[1] [PāGa1959, v. 72], [GSS1912, v. 218], [MaSi1910, ch. 15.45*cd*–46].

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1556) by Śaṅkara Vāriyar and Nārāyaṇa [Līlā1975].[2] In this study, I refer to Bhāskara's auto-commentary and the commentaries by Gaṇeśa and Nārāyaṇa.

2.1 Permutations of distinct digits

Rule 1 [Līlā1937, vol. 2, v. 261]:

स्थानान्तमेकादिचयाङ्कघातः संख्याविभेदा नियतैः स्युरङ्कैः ।
भक्तोऽङ्कमित्याङ्कसमासनिघ्नः स्थानेषु युक्तो मितिसंयुतिः स्यात् ॥

sthānāntamekādicayāṅkaghātaḥ saṃkhyāvibhedā niyataiḥ syuraṅkaiḥ |
bhakto'ṅkamityāṅkasamāsanighnaḥ sthāneṣu yukto mitisaṃyutiḥ syāt ||

The product of the digits (*aṅka*) beginning with one and increasing by one up to the [number of] places is the [number of] variations of numbers (*saṃkhyāvibheda*) with fixed digits. [The product] divided by the number of digits (*aṅkamiti*) and multiplied by the sum of the digits, and added in the places, will be the sum of the numbers (*mitisaṃyuti*).[3]

Example [Līlā1937, vol. 2, v. 262]:

द्विकाष्टकाभ्यां त्रिनवाष्टकैर्वा निरन्तरं द्व्यादिनवावसानैः ।
संख्याविभेदाः कति संभवन्ति तत्संख्यकैक्यानि पृथग्वदाशु ॥

dvikāṣṭakābhyāṃ trinavāṣṭakairvā nirantaraṃ dvyādinavāvasānaiḥ |
saṃkhyāvibhedāḥ kati saṃbhavanti tatsaṃkhyakaikyāni pṛthagvadāśu ||

How many are variations of numbers with two and eight? Or with three, nine and eight? Or with [the digits] from two to nine without interval? Tell promptly the sums of those numbers (*saṃkhyaikya*) separately.

2.2 Auto-commentary

1. Given digits: 2, 8. The number of variations is $1\cdot 2 = 2$. This 2 is multiplied by the sum of the two digits $2 + 8 = 10$. The product 20 is divided by 2, the number of digits. The result 10 is added in two places. The sum of the numbers 110 is obtained.
2. Given digits: 3, 9, 8. The number of variations is $1 \cdot 2 \cdot 3 = 6$. Then 6 is multiplied by the sum of digits $8 + 3 + 9 = 20$. The product is 120. The product is divided by 3, the number of digits. The quotient 40 is added in three places. The sum of the numbers is 4440.

[2] For the joint authorship of Śaṅkara and Nārāyaṇa's *Kriyākramakarī* see p. xvi.

[3] Bhāskara uses the term *miti* as a synonym of *saṃkhyā* (number).

3. Given digits: 2, 3, 4, 5, 6, 7, 8, 9. The number of variations is 40320. The sum of the numbers is 2463999975360.

2.3 Commentary by Gaṇeśa

Gaṇeśa refers to spreading (*prastāra*) and enumerates 28, 82 in the first case and 398, 389, 983, 938, 893, 839 in the second case.[4] He says 'the [method for] spreading the variations with digits in clay (*loṣṭa*) told by Nārāyaṇa and so on should be known'.[5]

[4] Bhāskara uses the term *prastāra* in *Līlāvatī* 115 for metrics, but he discusses only the number of combinations.

[5] लोष्टाङ्कैर्भेदप्रस्तारो नारायणादिभिरुक्तो ज्ञेयः ।
loṣṭāṅkair bhedaprastāro nārāyaṇādibhir ukto jñeyaḥ | (p. 276)

I can not find the term *loṣṭāṅka* in the *Kriyākramakarī*. I do not know whom Gaṇeśa referred to. Nārāyaṇa Paṇḍita, who wrote the *Gaṇitakaumudī* in 1356, used a term *laḍḍuka* for spreading in the chapter called *aṅkapāśa*. This is similar to *loṣṭakaprastāra* referred to by Bhaṭṭotpala for combination in his commentary on the *Bṛhatsaṃhitā* 76.22. Hayashi [Hay1979, p. 162] showed how to use clay-marks. Based on Hayashi's explanation I discuss how to arrange the marks in an example.

Arrange three digits from 1, 2, 3, 4, 5, 6, 7, 8. Draw two rows of cells, one of which is filled by digts from 1 to 8. First put three marks, whose number is equal to the number of digits. This is shown as

1	2	3	4	5	6	7	8
					0	0	0

This indicates an arrangement of digits: 678. Then move the mark under 6 to the place under 5:

1	2	3	4	5	6	7	8
				0		0	0

This is an arrangement of 578. In the same way move the mark to the left until it comes to the place under 1:

1	2	3	4	5	6	7	8
0						0	0

which indicates 178. When the mark comes to the place under 1 put marks under 5, 6 and 8:

1	2	3	4	5	6	7	8
				0	0		0

When some marks are used in this way the digits are arranged in ascending order only. [Hay1979, p. 162]; [Kus1993, pp. 496–500].

Commentary by Nārāyaṇa

Nārāyaṇa explains the operation 'added in the places' as follows: 'If there are two places, after one place the number at the place of ones one should again place the same (number) at the place of tens. This amounts to saying the following. If there are two places, one should multiply the number by eleven.'[6]

Note: Neither Bhāskara nor commentators show how to add the numbers. In his translation of the passage of the auto-commentary, Colebrooke [Col2005, pp. 123–124] explains it in square brackets as

$$\begin{array}{ll} [1\ 0 & \\ & 1\ 0] \end{array}$$

in the first case, and

$$\begin{array}{lll} [4\ 0 & & \\ & 4\ 0 & \\ & & 4\ 0] \end{array}$$

in the second case the number 10 may be set out as

$$\begin{array}{rr} & 1\ 0 \\ 1\ 0 & \end{array} \quad \text{or} \quad \begin{array}{ll} 1\ 0 & \\ & 1\ 0 \end{array}$$

in the first case and added together:

$$\begin{array}{r} 1\ 0 \\ 1\ 0\ \ \ \\ 1\ 1\ 0 \end{array} \qquad \text{or} \qquad \begin{array}{r} 1\ 0\ \ \ \\ 1\ 0 \\ 1\ 1\ 0 \end{array}$$

This operation is equivalent to $10 \times (1 + 10) = 110$. The process to calculate the sum is $\frac{2\times(2+8)}{2} \times 11 = 110$.

In the second case the calculation is

[6] यदि स्थानद्वयं भवति तदा एकस्थाने तां संख्यां विन्यस्य पुनर्दशस्थाने च ताम् एव संख्यां विन्यसेत्। एतदुक्तं भवति। यदि स्थानद्वयं तदा तां संख्याम् एकादशभिर्निहन्यात्।
yadi sthānadvayaṃ bhavati tadā ekasthāne tāṃ saṃkhyāṃ vinyasya punar daśasthāne ca tām eva saṃkhyāṃ vinyaset | etad uktaṃ bhavati | yadi sthānadvayaṃ tadā tāṃ saṃkhyām ekādaśabhir nihanyāt |[Līlā1975, p. 459].

$$\begin{array}{r} 40 \\ 40 \\ 40 \\ \hline 4440 \end{array} \quad \text{or} \quad \begin{array}{l} 40 \\ 40 \\ 40 \\ \hline 4440 \end{array}$$

The sum of the numbers is $40 \times (100 + 10 + 1) = 4440$.[7]

In the third case,

$\frac{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7 \cdot 8}{8}(2 + 3 + 4 + 5 + 6 + 7 + 8 + 9) = \frac{40320}{8} \times 44$. The result 221760 is added in 8 places.

$$\begin{array}{r} 221760 \\ 221760 \\ 221760 \\ 221760 \\ 221760 \\ 221760 \\ 221760 \\ 221760 \\ \hline 2463999975360 \end{array}$$

In general, among the 9 digits, n digits, $a_1, a_2, \ldots, a_n$, are taken. The number of variations is $P_n = n! = 1 \cdot 2 \cdots (n-2) \cdot (n-1) \cdot n$, where n is the number of distinct digits or places. A variation formed by concatenating numerals is a number. The process to calculate the sum of the numbers in stanza 261 implies the derivation of the rule. P_n divided by n and multiplied by the sum of the given numbers is added in each decimal place. Each a_i occurs $\frac{P_n}{n}$ times in each place. Therefore, the sum of the digits in each place is

$$\frac{P_n}{n} \times \sum_{i=1}^{n} a_i.$$

The operation that the result is added in the places is equivalent to multiplication by $\sum_{i=0}^{n-1} 10^i$. In the auto-commentary Bhāskara prescribes multiplication first and division after.

[7] यदि स्थानत्रयं तदा चन्द्ररुद्रैः तां संख्यां निहन्यात् ।
yadi sthānatrayaṃ tadā candrarudraiḥ tāṃ saṃkhyāṃ nihanyāt | [Līlā1975, p. 459].
If there are three places one should multiply the number by one hundred and eleven.

2.4 Another example

The first half of Rule 1 can be applied to the permutations of different objects whose number is more than 9 [Līlā1937, vol. 2, v. 263]:

> पाशाङ्कुशाहिडमरूककपालशूलैः खट्वाङ्गशक्तिशरचापयुतैर्भवन्ति ।
> अन्योन्यहस्तकलितैः कति मूर्तिभेदाः शम्भोर्हरेरिव गदारिसरोजशङ्खैः ॥
>
> *pāśāṅkuśāhiḍamarūkakapālaśūlaiḥ khaṭvāṅgaśaktiśaracāpayutairbhavanti |*
> *anyonyahastakalitaiḥ kati mūrtibhedāḥ śambhorharereiva gadārisarojaśaṅkhaiḥ ||*
>
> How many are the variations of the images of Śambhu when the noose, elephant's hook, serpent, tabor, skull, trident, together with club, crosier, arrow, and bow are held in one or the other of his hands? Similarly [how many are the variations of the images] of Hari with the club, discus, lotus and conch?

2.5 Auto-commentary

The images of Śambhu are 3628800 ($= 1\cdot2\cdot3\cdot4\cdot5\cdot6\cdot7\cdot8\cdot9\cdot10$) ways. Those of Hari are 24 ($= 1\cdot2\cdot3\cdot4$) ways.

Note: According to Sarma, the western Cālukyan monarch Someśvara III (1127-1138), a contemporary of Bhāskara, enumerates the twenty-four names in the *Mānasollāsa* 3.1.688-694 [Sar2006, pp. 115–126]. Sarma refers to certain *Purāṇas* which mention the twenty four images and names [Sar2006, p. 116].

3 Permutations of non-distinct digits

Bhāskara treats this rule as a special (*viśeṣa*) case of Rule 1.

Rule 2 [Līlā1937, vol. 2, v. 264]:

> यावत्स्थानेषु तुल्याङ्काः तद्भेदैस्तु पृथक्कृतैः ।
> प्राग्भेदा विहृता भेदाः तत्संख्यैक्यं च पूर्ववत् ॥
>
> *yāvatsthāneṣu tulyāṅkāḥ tadbhedaistu pṛthakkṛtaiḥ |*
> *prāgbhedā vihṛtā bhedāḥ tatsaṃkhyaikyaṃ ca pūrvavat ||*
>
> The variations [computed as] before, divided by the variations computed separately for as many places as there are the same digits, are the variations. The sum of these numbers is [calculated] as previously.

Example [Līlā1937, vol. 2, v. 265]:

> द्विद्व्येकभूपरिमितैः कति संख्यकाः स्युः
> तासां युतिश्च गणकाशु मम प्रचक्ष्व ।

अम्भोधिकुम्भिशरभूतशरैस्तथाङ्कैः
चेदङ्कपाशमितियुक्तिविशारदोऽसि ॥

dvidvyekabhūparimitaiḥ kati saṃkhyakāḥ syuḥ
tāsāṃ yutiśca gaṇakāśu mama pracakṣva |
ambhodhikumbhiśarabhūtaśaraistathāṅkaiḥ
cedaṅkapāśamitiyuktiviśārado'si ||

How many are the numbers (*saṃkhyaka*) with [the digits] two, two, one and one and their sum? Tell me quickly, mathematician, also with [the digits] four, eight, five, five and five, if you are conversant with the reasoning (*yukti*) with numbers (*miti*) in *aṅkapāśa*.

3.1 Auto-commentary

1. Given digits: 2,2,1,1.

   The number of variations is $\frac{4!}{(2!\cdot 2!)} = 6$. Here Bhāskara enumerates the variations shown as below and gives the sum of the numbers 9999.

2. Given digits: 4,8,5,5,5.

   The number of variations is $\frac{5!}{3!} = 20$. Here also Bhāskara enumerates the variations shown as below and gives the sum of the numbers 1199988.

3.2 Commentary by Gaṇeśa

Bhāskara does not explain how to calculate the sum of the numbers and says only 'as before' in the first case. According to Gaṇeśa, the sum of the numbers is obtained as follows: The number of variations 6 is multiplied by the sum of the given digits $(2+2+1+1=6)$. The product 36 is divided by the number of digits 4. The result 9 is added in four places; 9999. The sum of numbers in the second case is obtained as follows: $\frac{20\cdot(4+8+5+5+5)}{5} = 108$, which is added in five places: 1199988.

Gaṇeśa also enumerates the variations in the two cases. The enumerations in these cases are not identical (see Table 1). Here there is no rule for spreading.

Note: I think that the number to be added in the places can be calculated in an alternative way. Divide the number of variations by the number of digits. In the first case, divide 6 by 4. The quotient is 3/2. Because both the digits 2 and 1 are used twice, multiply 3/2 by 2. Thus 3 is obtained, that means 1 and 2 occur three times in each decimal place. Therefore the sum of digits in each

decimal place is $1 \cdot 3 + 2 \cdot 3 = 9$. In the second case 4 and 8 occur $20/5 = 4$ times, and 5 occurs $4 \cdot 3 = 12$ times in each place. Therefore the sum of digits in each decimal place is $4 \cdot 4 + 8 \cdot 4 + 5 \cdot 12 = 108$.

Table 1: Sum of numbers according to Bhāskara and Gaṇeśa.

Bhāskara	Gaṇeśa	Bhāskara	Gaṇeśa	Bhāskara	Gaṇeśa
2211	2211	48555	48555	45558	55584
2121	2112	84555	54855	85455	55548
2112	2121	54855	55485	85545	58545
1212	1221	58455	55458	85554	58554
1221	1212	55485	45855	54585	58455
1122	1122	55845	45585	58545	55845
		55548	85455	55458	54558
		55584	84555	55854	45558
		45855	54585	54558	85545
		45585	55854	58554	85554

4 Permutations of distinct digits from 1 to 9

Rule 3 [Līlā1937, vol. 2, v. 266]:

> स्थानान्तमेकापचितान्तिमाङ्कघातोऽसमाङ्कैश्च मितिप्रभेदाः ॥
>
> *sthānāntamekāpacitāntimāṅkaghāto'sa māṅkaiśca mitiprabhedāḥ* ||
>
> The product of [the numbers beginning with] the last digit (*antimāṅka*) (i.e. 9) and [successively] decreasing by one until the [given] place, is the variations of numbers (*mitiprabheda*) with dissimilar digits.

Among 9 digits, n digits are taken and arranged in a certain order. The number of variations is ${}_9P_n = 9 \cdot 8 \cdots (9 - (n-1))$.

Example [Līlā1937, vol. 2, v. 267]:

> स्थानषट्कस्थितैरङ्कैः अन्योन्यं खेन वर्जितैः ।
> कति संख्याविभेदाः स्युः यदि वेत्सि निगद्यताम् ॥
>
> *sthānaṣaṭkasthitairaṅkaiḥ anyonyaṃ khena varjitaiḥ* |
> *kati saṃkhyāvibhedāḥ syuḥ yadi vetsi nigadyatām* ||
>
> How many are the variations of numbers with digits [different] from each other except zero in six places? Tell if you know.

4.1 Auto-commentary

Bhāskara calculates the number of variations $9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 = 60480$.

4.2 Commentary by Gaṇeśa

Gaṇeśa enumerates some of the numbers which occupy two decimal places. He does not list $9 \cdot 8 = 72$ numbers, but enumerates 12, 13, 14, 15, 16, 17, 18, 19, 21, 23, 24, 25, 26, 27, 28. Then he enumerates numbers which occupy three decimal places: 123, 124, 125, 126, 127, 128, 129, 136, 137, 138, 132, 134, 135, and gives the number of variations as 504.

Next he gives the *upapatti* as follows. 'From ten to ninety-nine there are nine tens [of numbers] having two places. Among them nine digits [namely], ten and twenty, etc. are accompanied by zero. Nine [numbers, namely] eleven and twenty-two etc. have equal digits. Thus nine multiplied by ten minus two (8) is the variations of numbers in two places 72. Similarly from one hundred to nine hundred and ninety-nine there are numbers in three places. Among them also, one hundred 100, one hundred and ten 110, etc. are accompanied by zero. One hundred and eleven 111, one hundred and twenty-two 122, etc. also have two equal digits. Therefore the variations in two places 72 multiplied by ten minus three (7) is the variations of numbers in three places 504.'[8]

Note: In the case of the numbers occupying two places, I summarize Gaṇeśa's explanation. There are nine rows. In each row there are eight numbers. Therefore the number of variations is $9 \cdot 8 = 72$.

[8] दशारभ्य नवनवतिपर्यन्तं स्थानद्वयाङ्कानां नव दशकाः। तेषु दशविंशत्यादयो नवाङ्काः शून्यसमेताः। एकादशद्वाविंशत्यादयो नव तुल्याङ्काः। एवं द्व्यूनदशकेन ८ गुणिता नव स्थानद्वयाङ्कभेदाः ७२ स्युः। एवं शतमारभ्यैकोनसहस्रपर्यन्तं स्थानत्रयाङ्काः। तेष्वपि शत १०० दशोत्तरशतादयः ११० शून्यसमेताः। एकादशोत्तरशत १११ द्वाविंशत्युत्तरादिष्वपि १२२ द्वौ समाङ्कौ स्तः। अतः त्र्यूनदशकेन ७ गुणिताः स्थानद्वयभेदाः ७२ स्थानत्रयाङ्कभेदाः ५०४ स्युः।

daśārabhya navanavatiparyantaṃ sthānadvayāṅkānāṃ nava daśakāḥ | *teṣu daśaviṃśatyādayo navāṅkāḥ śūnyasametāḥ* | *ekādaśadvāviṃśatyādayo nava tulyāṅkāḥ* | *evaṃ dvyūnadaśakena 8 guṇitā nava sthānadvayāṅkabhedāḥ 72 syuḥ* | *evaṃ śatam ārabhyaikonasahasraparyantaṃ sthānatrayāṅkāḥ* | *teṣv api śata 100 daśottaraśatādayaḥ 110 śūnyasametāḥ* | *ekādaśottaraśata 111 dvāviṃśatyuttarādiṣv api 122 dvau samāṅkau staḥ* | *ataḥ tryūnadaśakena 7 guṇitāḥ sthānadvayabhedāḥ 72 sthānatrayāṅkabhedāḥ 504 syuḥ* | (pp. 281–282.)

$\boxed{10}$	$\boxed{11}$	12	13	14	15	16	17	18	19
$\boxed{20}$	21	$\boxed{22}$	23	24	25	26	27	28	29
$\boxed{30}$	31	32	$\boxed{33}$	34	35	36	37	38	39
$\boxed{40}$	41	42	43	$\boxed{44}$	45	46	47	48	49
$\boxed{50}$	51	52	53	54	$\boxed{55}$	56	57	58	59
$\boxed{60}$	61	62	63	64	65	$\boxed{66}$	67	68	69
$\boxed{70}$	71	42	73	74	75	76	$\boxed{77}$	78	79
$\boxed{80}$	81	82	83	84	85	86	87	$\boxed{88}$	89
$\boxed{90}$	91	92	93	94	95	96	97	98	$\boxed{99}$

In the case of the numbers occupying three places, number 12, for example, can be connected with seven digits 3, 4, 5, 6, 7, 8, 9. Similarly 13 can be connected with seven digits 2, 4, 5, 6, 7, 8, 9. Therefore the number of variations in three places is $(9 \cdot 8) \cdot 7$.

$\boxed{100}$	$\boxed{101}$	$\boxed{102}$	$\boxed{103}$	$\boxed{104}$	$\boxed{105}$	$\boxed{106}$	$\boxed{107}$	$\boxed{108}$	$\boxed{109}$
$\boxed{110}$	$\boxed{111}$	$\boxed{112}$	$\boxed{113}$	$\boxed{114}$	$\boxed{115}$	$\boxed{116}$	$\boxed{117}$	$\boxed{118}$	$\boxed{119}$
$\boxed{120}$	$\boxed{121}$	$\boxed{122}$	123	124	125	126	127	128	129
$\boxed{130}$	$\boxed{131}$	132	$\boxed{133}$	134	135	136	137	138	139
$\boxed{140}$	$\boxed{141}$	142	143	$\boxed{144}$	145	146	147	148	149
$\boxed{150}$	$\boxed{151}$	152	153	154	$\boxed{155}$	156	157	158	159
$\boxed{160}$	$\boxed{161}$	162	163	164	165	$\boxed{166}$	167	168	169
$\boxed{170}$	$\boxed{171}$	172	173	174	175	176	$\boxed{177}$	178	179
$\boxed{180}$	$\boxed{181}$	182	183	184	185	186	187	$\boxed{188}$	189
$\boxed{190}$	$\boxed{191}$	192	193	194	195	196	197	198	$\boxed{199}$

5 Permutations of digits that add up to a specified sum

Rule 4 [Līlā1937, vol. 2, vv. 268–269]:

निरेकमङ्कैक्यमिदं निरेकस्थानान्तमेकापचितं विभक्तम् ।
रूपादिभिस्तन्निहतैः समाः स्युः संख्याविभेदा नियतेऽङ्कयोगे ॥
नवान्वितस्थानकसंख्यकाया ऊनेऽङ्कयोगे कथितं तु वेद्यम् ।
संक्षिप्तमुक्तं पृथुताभयेन नान्तोऽस्ति यस्माद्गणितार्णवस्य ॥

nirekamaṅkaikyamidaṃ nirekasthānāntamekāpacitaṃ vibhaktam |
rūpādibhistannihataiḥ samāḥ syuḥ saṃkhyāvibhedā niyate'ṅkayoge ||
navānvitasthānakasaṃkhyakāyā ūne'ṅkayoge kathitaṃ tu vedyam |
saṃkṣiptamuktaṃ pṛthutābhayena nānto'sti yasmādgaṇitārṇavasya ||

> The sum of digits less one [and successively] decreasing by one until the [given] place less one is divided by one and so on. The product of these is equal to the variations of the number when the sum of digits is fixed. It is to be known that this was told for the case when the sum of digits is less than the number of places added to nine. Only the compendium was told for fear of prolixity (*pṛthutābhayena*)[9] since the ocean of calculation has no bounds.

This is the case when the numbers from 1 to 9 are arranged so as to give the given sum (s) in a given number of places (n). The number of variations is:[10]

$$\frac{s-1}{1} \cdot \frac{s-2}{2} \cdots \frac{s-(n-1)}{n-1} (n+9<s).$$

Example [Līlā1937, vol. 2, v. 270]:

> पञ्चस्थानस्थितैरङ्कैर्यद्यद्योगस्त्रयोदश ।
> कति भेदा भवेत्संख्या यदि वेत्सि निगद्यताम् ॥
>
> *pañcasthānasthitairaṅkair yadyadyogastrayodaśa |*
> *kati bhedā bhavetsaṃkhyā yadi vetsi nigadyatām ||*
>
> How many are the variations of numbers with digits in five places, the sum of which is thirteen? Tell if you know.

5.1 Auto-commentary

The number of variations is $\frac{12}{1} \cdot \frac{11}{2} \cdot \frac{10}{3} \cdot \frac{9}{4} = 495$.

5.2 Commentary by Gaṇeśa

Gaṇeśa counts variations of each of the eighteen cases, which is calculated by Rule 2.

---

[9] In the [Līlā1937, vol. 1, v. 114] on combination also he uses a similar phrase 'for fear of prolixity' (*vistṛter bhāyāt*).

[10] In [Yan1980, p. 371] Hayashi in footnote 37 shows a derivation of the rule. Also see [Gup1995, pp. 11–12].

Sr. No.	Number	Ways	Sr. No.	Number	Ways
1	11119	5	10	12343	60
2	11128	20	11	22225	5
3	11137	20	12	12226	20
4	11146	20	13	33331	5
5	11155	10	14	44221	30
6	11236	60	15	12235	60
7	11245	60	16	33322	10
8	11344	30	17	22234	20
9	11335	30	18	72211	30

Note: $s = 13$. $n = 5$. This is an example of the partition of 13 in five numbers. $x_1 + x_2 + x_3 + x_4 + x_5 = 13$. $1 \leq x_i \leq 9$ (x_i is integer).

Neither Bhāskara nor the two commentators mention the sum of the numbers. I calculate the sum in the same way as before.

Cases (1) (11) (13): The number of variations is obtained by $5!/4!$. By Rule 2, $(5 \cdot 13)/5 = 13$. And 13 is added in five places. The sum of the numbers is 144443.

Cases (2) (3) (4) (12) (17): The number of variations is obtained by $5!/3!$. By Rule 2, $(20 \cdot 13)/5 = 52$. 52 is added in five places. The sum of the numbers is 577772.

Cases (5) (16): The number of variations is obtained by $5!/(3!2!)$. By Rule 2, $(10 \cdot 13)/5 = 26$. 26 is added in five places. The sum of the numbers is 288886.

Cases (6) (7) (10) (15): The number of variations is obtained by $5!/2!$. By Rule 2, $(60 \cdot 13)/5 = 156$. 156 is added in five places. The sum of the numbers is 1733316.

Cases (8) (9) (14) (18): The number of variations is obtained by $5!/(2!2!)$. By Rule 2, $(30 \cdot 13)/5 = 78$. 78 is added in five places. The sum of the numbers is 866658.

Then their totals in each case are added together: $144443 \times 3 = 433329$, $577772 \times 5 = 2888860$, $288886 \times 2 = 577772$, $1733316 \times 4 = 6933264$, and $866658 \times 4 = 3466632$. The sum of all is 14299857.

Or the sum of the numbers can be calculated by the method mentioned in Rule 1. The number of variations is 495. The sum of digits is 13. The number of places is 5. Then $\frac{495 \times 13}{5} = 1287$. The result 1287 is added in five places.

$$\begin{array}{r}
1\,2\,8\,7\\
1\,2\,8\,7\\
1\,2\,8\,7\\
1\,2\,8\,7\\
1\,2\,8\,7\\
\hline
1\,4\,2\,9\,9\,8\,5\,7
\end{array}$$

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Part III

The Bījagaṇita

उत्पादकं यत्प्रवदन्ति बुद्धेः
अधिष्ठितं सत् पुरुषेण सांख्याः ।
व्यक्तस्य कृत्स्नस्य तदेकबीजं
अव्यक्तमीशं गणितं च वन्दे ॥

utpādakaṃ yat pravadanti buddher
adhiṣṭhitaṃ sat puruṣeṇa sāṃkhyāḥ |
vyaktasya kṛtsnasya tad ekabījam
avyaktam īśaṃ gaṇitaṃ ca vande ||

I revere that unmanifest supreme power (*avyakta-īśa*), whom seers (*sāṃkhyās*) [who have understood the nature of the self] declare to be the producer of clear intellect (*buddhi*), and [also] as the sole cause (*bīja* or seed) for all that is manifest (*vyakta*), being so explained by a holy person.

I [simultaneously] revere that mathematics in its unmanifest form (*avyakta-gaṇita*, namely algebra), which the mathematicians (*sāṃkhyās*) declare to be the generator of clear [mathematical] thinking (*buddhi*), and [also] as the sole cause (*bīja* or seed) for all that is [dealt with] in arithmetic (*vyakta*), being so expounded by an able mathematician.

The *Bījagaṇita* of Bhāskarācārya: Some highlights

Sita Sundar Ram*

1 Introduction

Arithmetic and algebra form two fundamental branches of mathematics. While arithmetic deals with mathematical operations, algebra deals with the determination of unknown entities. Hence arithmetic was known as *vyakta* (manifest) *gaṇita* and algebra was known as *avyakta* (unmanifest) *gaṇita.* Starting from the earliest extant treatises like the *Śulbasūtras,* Indian mathematicians like Āryabhaṭa, Brahmagupta, Mahāvīra, Śrīpati, Śrīdhara, Padmanābha and others have contributed directly and indirectly to the growth of algebra. Bhāskara is no exception. In addition to his famous treatise on arithmetic, the *Līlāvatī*, Bhāskara has devoted an entire treatise on algebra called the *Bījagaṇita*.

Explaining why algebra has been called *Bījagaṇita*, Bhāskara observes [BīGa1927, p. 99]:

बीजं मतिर्विविधवर्णसहायिनी हि
मन्दावबोधविधये विबुधैर्निजाऽद्यैः ।
विस्तारिता गणकतामरसांशुमद्भिः
या सैव बीजगणिताह्वयतामुपेता ॥

bījaṃ matirvividhavarṇasahāyinī hi
mandāvabodhavidhaye vibudhairnijā'dyaiḥ |
vistāritā gaṇakatāmarasāṃśumadbhiḥ
yā saiva bījagaṇitāhvayatāmupetā ||

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> Intelligence alone is algebra. Indeed the variety of symbols are its associates. Ancient teachers, enlightening mathematicians as the sun irradiates the lotus, have in the main displayed their intelligence for facilitating the understanding of the dull-witted. That has now attained the name *Bījagaṇita*.

He also remarks [BīGa1927, p. 127]:

> उपपत्तियुतं बीजगणितं गणका जगुः ।
> न चेदेवं विशेषोऽस्ति न पाटीबीजयोर्यतः ॥
>
> *upapattiyutaṃ bījagaṇitaṃ gaṇakā jaguḥ |*
> *na cedevaṃ viśeṣo'sti na pāṭībījayoryataḥ ||*
>
> Mathematicians have declared algebra to be calculation accompanied by proofs; otherwise, there would be no distinction between arithmetic and algebra.

2 An overview of the topics discussed in the *Bījagaṇita*

In Bhāskara's *Bījagaṇita,* the initial chapters deal with *dhanarṇaṣaḍvidha* (rules of postive and negative signs), *khaṣaḍvidha* (rules of zero and infinity) *avyakta-ṣaḍvidha* (operations of unknowns), *karaṇī* (surds), *kuṭṭaka* (indeterminate equations of the first degree), *vargaprakṛti* and *cakravāla* (indeterminate equations of the second degree). In the later sections, Bhāskara teaches the applications of the *sūtras* stated earlier in his text.

It may also be mentioned that Bhāskara was the first to classify the *Bījagaṇita* into two parts namely, the tools of algebra and its applications. At the end of the first section after the chapter on *Vargaprakṛti* and *Cakravāla,* he says [BīGa1927, p. 43]:

> उक्तं बीजोपयोगीदं संक्षिप्तं गणितं किल ।
> अतो बीजं प्रवक्ष्यामि गणकानन्दकारकम् ॥
>
> *uktaṃ bījopayogīdaṃ saṃkṣiptaṃ gaṇitaṃ kila |*
> *ato bījaṃ pravakṣyāmi gaṇakānandakārakam ||*
>
> The section of this science of calculation which is essential for analysis has been briefly set forth. Next I shall propound analysis, which is the source of pleasure to the mathematician.

Among the commentaries of the *Bījagaṇita,* the *Bījapallava* of Kṛṣṇa Daivajña and the *Sūryaprakāśa* of Sūryadāsa are the most popular. In what follows, based on our study of the *Bījapallava,* in the light of these commentaries, we discuss some of its special features.

2.1 Positive and negative numbers

Dhanarṇaṣaḍvidha explains the six fundamental operations dealing with positive and negative numbers. They are *saṅkalana, vyavakalana, guṇana, bhajana, varga* and *vargamūla*. Brahmagupta (b. 598 CE) was the first to detail the operations involving positive and negative numbers. It should be mentioned here that Kṛṣṇa Daivajña in his commentary *Bījapallava* [BīPa2012, p. 19] on the *Bījagaṇita* explains the addition and subtraction of negative and positive numbers with the help of identifying them on a straight line. Modern school texts call this the "number line".

About negative numbers, Bhāskara says in [BīGa1927, p. 2]:

यानि ऋणगतानि तानि ऊर्ध्वबिन्दूनि च ।

yāni ṛṇagatāni tāni ūrdhvabindūni ca |

Numbers which are negative also get indicated by a dot on top [of the number].

He further adds [BīGa1927, p. 4]:

न मूलं क्षयस्यास्ति तस्याकृतित्वात् ।

na mūlaṃ kṣayasyāsti tasyākṛtitvāt |

A negative number has no square root because of its non-square nature.

2.2 Zero and infinity

The discovery of zero not only helped to establish the place value system but also to distinguish between negative and positive numbers. Though Brahmagupta was the first to deal with operations of zero, it was Bhāskara who first explained with a beautiful simile that any number divided by zero results in a quantity whose magnitude does not change whatsoever, just as in the case of mathematical infinity. He says in the chapter [BīGa1927, p. 6]:

अस्मिन् विकारः खहरे न राशौ अपि प्रविष्टेष्वपि निःसृतेषु ।
बहुष्वपि स्याल्लयसृष्टिकालेऽनन्तेऽच्युते भूतगणेषु यद्वत् ॥

asmin vikāraḥ khahare na rāśau api praviṣṭeṣvapi niḥsṛteṣu |
bahuṣvapi syāllayasṛṣṭikāle'nante'cyute bhūtagaṇeṣu yadvat ||

Just as at the period of destruction or creation of the worlds, though numerous orders of beings are absorbed or put forth, there is no change in the Infinite and Immutable God, so in the quantity which has zero for its divisor, there is no change, though many may be added or subtracted.

2.3 Karaṇī

The section on *Karaṇī* or surds is dealt with in detail by Bhāskara. He says [BīGa1927, p. 12]:

योगं करण्योर्महतीं प्रकल्प्य घातस्य मूलं द्विगुणं लघुं च ।
योगान्तरे रूपवदेतयोस्ते वर्गेण वर्गं गुणयेत् भजेच्च ॥

yogaṃ karaṇyormahatīṃ prakalpya ghātasya mūlaṃ dviguṇaṃ laghuṃ ca|
yogāntare rūpavadetayoste vargeṇa vargaṃ guṇayet bhajecca ||

Term the sum of two irrationals the great surd; and twice the square root of their product, the less one. The sum and difference of these reckoned like integers are so [of the original roots]. Multiply and divide a square by a square [Col2005, p. 145].

The sum of two numbers under the square root sign is denoted by M (*mahatī*). Twice the square root of their product is denoted by L (*laghu*). The sum and difference of the two surds are respectively $\sqrt{M+L}$ and $\sqrt{M-L}$. If a number is to be multiplied or divided by a given number, multiply or divide the given number under the radical sign by the square of the given number.

In other words, $a+b=M; \quad 2\sqrt{ab}=L$. Then,

$$\left(\sqrt{a}+\sqrt{b}\right)^2 = a+b \pm 2\sqrt{ab} \Rightarrow \sqrt{a} \pm \sqrt{b} = \sqrt{M \pm L}.$$

While dealing with division of surds, Bhāskara has skillfully used what is called rationalization of denominators in modern mathematics. In the example given the divisor is $\sqrt{27}-\sqrt{25}$. The student is then asked to multiply and divide by $\sqrt{27}+\sqrt{25}$ to get the result with integers in the denominator.

2.4 Kuṭṭaka: Indeterminate equation of the first degree

Beginning with Āryabhaṭa, most of the ancient Indian mathematicians dealt with the *kuṭṭaka* or pulverizer method to solve indeterminate equations of the first degree, $ax+c=by$. After giving the definition of the word *kuṭṭaka* and explaining the method, Bhāskara has considered various forms of *kuṭṭaka* with negative and positive dividend a (*bhājya*), divisor b (*bhājaka*) and additive c (*kṣepa*). Bhāskara also uses the *kuṭṭaka* method to solve equations with several unknowns. There are slight differences between the methods of Bhāskara, and those of Āryabhaṭa I, Brahmagupta, Mahāvīra, Āryabhaṭa II and Śrīpati.

The example given by Bhāskara is to obtain integer solutions for x and y in the equation: $221x+65=195y$. This equation gets reduced to $17x+5=15y$

after dividing throughout by 13. By using the *kuṭṭaka* method it can be shown that the least solution arrived at is the multiplier x (*guṇa*) $= 5$ and quotient y (*labdhi*) $= 6$.

Once we obtain the least solution, the procedure to get infinite solutions has been stated by Bhāskara as follows [BīGa1927, p. 26]:

> इष्टाहतस्वस्वहरेण युक्ते ते वा भवेतां बहुधा गुणाप्ती ।
>
> *iṣṭāhatasvasvahareṇa yukte te vā bhavetāṃ bahudhā guṇāptī* |
>
> The multiplier and quotient, being added to their respective abrading divisors multiplied by assumed numbers, become manifold [Col2005, pp. 161–162].

If $(x, y) = (5, 6)$ be the least solution, then infinite number of solutions can be generated using the following equations

$$x = 15t + 5 \qquad and \qquad y = 17t + 6,$$

where t is any integer.

In his elaborate treatment of *kuṭṭaka,* Bhāskara seems to have noticed that by the rule of some earlier writer, errors would arise in the case in which the dividend is negative. The problem arises when the *bhājya* and *kṣepa* are of opposite signs.

Bhāskara wonders [BīGa1927, p. 29]:

> भाज्ये भाजके वा ऋणगते परस्परभजनात् लब्धय ऋणगताः स्थाप्या इति किं तेन प्रयासेन ।
>
> *bhājye bhājake vā ṛṇagate parasparabhajanāt labdhaya ṛṇagatāḥ sthāpyā iti kim tena prayāsena* |
>
> If either the dividend or divisor become negative, the quotients of reciprocal division, would be stated as negative; which is a needless trouble [Col2005, p. 164].

Before we move on to next topic we may briefly mention about *sthirakuṭṭaka* and *saṃśliṣṭakuṭṭaka.* The simple indeterminate equation $ax \pm 1 = by$ is solved in the same way as $ax \pm c = by$ and is only a special case of the latter. It is called *sthirakuṭṭaka* and is used in astronomical calculations. The last topic dealt with in the *kuṭṭaka* chapter is the *saṃśliṣṭakuṭṭaka* or simultaneous linear Diophantine equation.

2.5 Vargaprakṛti and Cakravāla

At least from the time of Brahmagupta (628 CE), mathematicians in India were attempting the harder problem of solving equations of the second degree. In his *Brāhmasphuṭasiddhānta,* Brahmagupta gave a partial solution to the problem

of solving $Nx^2 + 1 = y^2$. The fundamental step in Brahmagupta's method for the general solution in positive integers of the equation $Nx^2 + 1 = y^2$ where N is any non-square integer, is to consider two auxiliary equations

$$Nx^2 + k_i = y^2, \qquad i = 1, 2$$

with k_i being chosen from $k_i = \pm 1, \pm 2, \pm 4$. A procedure known as *bhāvanā,* applied repeatedly, wherever necessary, helps us in deriving at least one possible solution of the original *vargaprakṛti* viz. $Nx^2 + 1 = y^2$. Using the *bhāvanā,* an infinite number of solutions can be obtained. Brahmagupta could find this auxiliary equation only by trial and error.

Let the general equations be $Nx_1^2 \pm k_1 = y_1^2$; $\quad Nx_2^2 \pm k_2 = y_2^2$. By the *bhāvanā* technique,

prakṛti	*kaniṣṭha*	*jyeṣtha*	*kṣepa*
N	x_1	y_1	k_1
	x_2	y_2	k_2

Then by *vajrābhyāsa* (cross multiplication) and direct multiplication, the new roots will be

$$x = x_1 y_2 + x_2 y_1, \qquad y = N x_1 x_2 + y_1 y_2, \qquad k = k_1 k_2.$$

Remarkable success was achieved by Bhāskara when he introduced a simple method to derive the auxiliary equation. This equation would have the required *kṣepas* $\pm 1, \pm 2, \pm 4$, simultaneously with two integral solutions from any auxiliary equation empirically formed with any simple value of the *kṣepa.* This method is the famous *cakravāla* or cyclic method, so named for its iterative character.

According to K. S. Shukla [Shu1954], the earliest author who refers to *cakravāla* is Udayadivākara (eleventh century CE), who quotes Jayadeva's verses on *cakravāla.* There is no mention of him by any other author. Bhāskara only says:

चक्रवालं जगुः ।

cakravālaṃ jaguḥ |

They called it *cakravāla.*

Note: The verbal form *jaguḥ* used here implies that the method was known prior to him.

2.5.1 Bhāskara's *cakravāla*

Bhāskara's *cakravāla* is based on the following lemma: If $Na^2 + k = b^2$ is an auxiliary equation where a, b, k are integers, k being negative or positive then,

$$N\left\{\frac{am+b}{k}\right\}^2 + \frac{m^2-N}{K} = \left\{\frac{bm+Na}{K}\right\}^2$$

where m is any whole number. The rationale is simple. Consider the equations:

$$Na^2 + k = b^2,$$
$$\text{and} \quad N(1)^2 + m^2 - N = m^2.$$

Using Brahmagupta's *bhāvanā* we have:

$$N(am+b)^2 + k(m^2-N) = (Na+mb)^2.$$

Dividing by k^2 we get,

$$N\left\{\frac{am+b}{k}\right\}^2 + \frac{m^2-N}{k} = \left\{\frac{bm+Na}{N}\right\}^2.$$

Here, m has to be so chosen such that $\frac{am+b}{k}$ is an integer since its value can be determined by means of the *kuṭṭaka, viz.,* by solving the equation $ax+b = ky$ for an integer solution and taking the solution for x as m. Obviously there can be an infinite number of values for m. But Bhāskara says m should be chosen so as to make $|m^2-N|$ minimum. If $\frac{m^2-N}{k}$ is equal to $\pm1, \pm2, \pm4$, then Brahmagupta's *bhāvanā* can be applied immediately. If $\frac{m^2-N}{k}$ is not one of the above values, then the *kuṭṭaka* is performed again and again till $\frac{m^2-N}{k}$ is equal to $\pm1, \pm2, \pm4$. It is possible that Bhāskara was aware that this process will end after a finite number of steps.

Kṛṣṇa Daivajña's valuable contribution in this chapter is that he has given the formula to find the new greater root [BīPa1958, p. 142].

> अन्यथापि ज्येष्ठापेक्षा चेत्तदा गुणकगुणितं ज्येष्ठं प्रकृतिगुणेन कनिष्ठेन युतं क्षेपभक्तं ज्येष्ठं भवतीत्यस्मदुक्तमार्गेण ज्येष्ठं कुर्यात्।
>
> *anyathāpi jyeṣṭhāpekṣā cettadā guṇakaguṇitaṃ jyeṣṭhaṃ prakṛtiguṇena kaniṣṭhena yutaṃ kṣepabhaktaṃ jyeṣṭhaṃ bhavatītyasmaduktamārgeṇa jyeṣṭhaṃ kuryāt* |
>
> Otherwise to find the greater root, the original greater root (b) multiplied by the multiplier (m) is added to the lesser root (a) multiplied by *prakṛti* (N) and the sum is divided by *kṣepa* (k).

This is nothing but the square root on the right hand side $\frac{bm+Na}{k}$. This method is especially useful while computing the square root of large numbers.

In the infinite system of values, there should be a set of two integers, one less than $\sqrt{N}$, the other greater than in the neighbourhood of $+\sqrt{N}$ and similarly N in the immediate two in the immediate neighbourhood of $\sqrt{N}$. The squares of these four integers are evidently nearer to N than the squares of any other value of m in the equation $am+b = ka_1$. According to Bhāskara, we have to choose that m which is closest to N.

In this connection, A. A. K. Ayyangar [Ayy1929, p. 235] makes a note that this rule has exceptions: "An exceptional case may occur when the root corresponding to the nearest square leads back to the previous step in the process of reduction. In this case, the root corresponding to the nearest of the remaining squares should be chosen." This exceptional case has not been explicitly noted by Bhāskara. However, the following example will make it clear.

Consider the equation $58x^2 + 1 = y^2$. The auxiliary equation is

$$58(1)^2 + 6 = (8)^2.$$

The corresponding *kuṭṭaka* equation is $x+8 = 6y$. The solutions for the above *kuṭṭaka* are (4, 2), (10, 3), (16, 4) and so on.

Choose (10, 3) which gives the least value for $\frac{m^2-N}{k}$ i.e., $\frac{|10^2-58|}{6} = 7$. The pair (10, 3) is chosen because by choosing (4, 2) we go back to the same *kuṭṭaka* and eventually arrive at $m = 10$.

Now take $x = 3$ and the equation is $58(3)^2 + 7 = 23^2$.

The corresponding *kuṭṭaka* equation is $3x + 23 = 7y$ following the same procedure as in the first step. Solution sets for this *kuṭṭaka* are (4, 5), (11, 8) and so on. Choosing (4, 5), we have *kṣepa* $= -6$ and the next x is 5 and the corresponding equation becomes

$$58(5)^2-6 = 38^2.$$

From the next *kuṭṭaka,* $5x + 38 = 6y$, we get solution sets (2, 8), (8, 13) and so on. Choosing (8, 13) as the least value for $\frac{m^2-N}{k}$, we get

$$58(13)^2 - 8.1 = 99^2.$$

Using *bhāvanā* with the columns,

prakṛti	*kaniṣṭha*	*jyeṣṭha*	*kṣepa*
58	13	99	-1
	13	99	-2

We have

$$\begin{aligned} x &= 2 \times 13 \times 99 = 2574 \\ y &= 58 \times 13 \times 13 + 99 \times 99 = 19603 \\ \text{and} \quad k &= -1 \times -1 = 1. \\ \text{Hence,} \quad & 58(2574)^2 + 1 = (19603)^2. \end{aligned}$$

An interesting anecdote

There is an interesting anecdote [Ste2000, p. 318] associated with equations of the type $Nx^2 + 1 = y^2$. In 1657, the famous French mathematician Fermat sent a public challenge to his friend, Bernard Frenicle de Bessy and then on to Brouncker and Wallis in England to get integral solution for the equation $61x^2 + 1 = y^2$. "We await", he challenged, "the solutions which, if England or Belgian and Celtic Gauls cannot give them, Narbonian Gaul will..."(meaning himself). None of them succeeded in solving the equation. It was only in 1732 that the renowned mathematician Euler gave a complete solution.

But remarkably, the very same equation had been dealt with and solved in a few steps by Bhāskara using the famous *cakravāla* method more than five centuries earlier. Bhāskara gave the least solution as $x = 226153980$ and $y = 1766319049$. No wonder Andre Weil [Wei1984, p. 81] exclaims "what would have been Fermat's astonishment if some missionary back from India had told him that his problem had been successfully tackled there by native mathematicians almost six centuries earlier."

Later, in the eighteenth century, Euler gave Brahmagupta's Lemma and its proof. He was aware of Brouncker's work on the above equation as presented by Wallis, but he was totally unaware of the contribution of the Indian mathematicians. He gave the basis for the continued fraction approach to solving the above equation which was put into a polished form by Lagrange in 1766. Euler was also responsible for wrongly naming the equation as "Pell's equation" thinking that the major contributions which Wallis had reported on as due to Brouncker, were in fact the work of Pell. Lagrange published his *Additions to Euler's Elements of Algebra* in 1771, and this contains his rigorous version of Euler's continued fraction approach to Pell's equation.

Selenius' observations

The following observations have been made by C. O. Selenius, in his special study on *Cakravāla* [Sel1975, pp. 177–178]:

1. *Cakravāla* represents the shortest possible continued fraction algorithm, with a minimum number of cycles.[1]
2. *Cakravāla* always produces the least (positive) solution, with or without the use of the *bhāvanā* technique from which any number of solution can be obtained.
3. Incidentally, *Cakravāla* also avoids large numbers in the calculations.

The equation $Nx^2 - 1 = y^2$

Bhāskara has also dealt with the case $Nx^2 - 1 = y^2$ i.e., when the additive or *kṣepa* is negative unity (-1). He says [BīGa1927, p. 40]:

> रूपशुद्धौ खिलोद्दिष्टं वर्गयोगो गुणो न चेत्।
>
> *rūpaśuddhau khiloddiṣṭaṃ vargayogo guṇo na cet* |
>
> [When the *kṣepa*] is negative unity, the solution of the problem is impossible unless the multiplier (*guṇa* = *prakṛti*), is the sum of the squares [of two numbers].

It is not clear whether the European mathematicians who later developed Pell's equation were aware of the above observation. A well-known result relating to convergents shows that particular solutions for $Nx^2 + 1 = y^2$ can always be found. But particular solutions for $Nx^2 - 1 = y^2$ are found only when the period of the continued fraction for $\sqrt{N}$ is odd. In addition, not all equations of this type can be solved [Old1963, pp. 115–117].

In this connection, we have the following known results of Olds on existence and non-existence of solutions for $Nx^2 - 1 = y^2$.

Result 1. If $N - 3$ is an integral multiple of 4, i.e., $N = 4k + 3$ for some natural number k, the equation $Nx^2 - 1 = y^2$ has no solutions.

Result 2. If N is a prime number of the form $4k + 1$, then the equation $Nx^2 - 1 = y^2$ always has solutions. This result is closely connected with a famous theorem stated by Fermat in 1640, and proved by Euler in 1754.

Theorem: Every prime p of the form $4k + 1$ can be expressed as the sum of two squares, and the representation is unique. That is, there exists one and only one pair of integers P, Q such that $p = P^2 + Q^2$.

[1] Naturally, the use of the *bhāvanā* technique shortens the calculations.

The above results conform to what Bhāskara had said five hundred years earlier. One of the examples considered by Bhāskara *viz.,* $8x^2 - 1 = y^2$ does not produce integral roots. This has been dealt with in detail in [BīPa2012, pp. 107–109].

Variations of *Vargaprakṛti*

So far, we discussed Bhāskara's *cakravāla* method to arrive at integral solution for vargaprakṛti problems of the type $Nx^2 - 1 = y^2$. In a few more forms of *vargaprakṛti,* Bhāskara does not insist on integral solutions, and hence does not apply the *cakravāla* method. These are explained by Kṛṣṇa Daivajña in detail [BīPa1958, pp. 149–154]. He considers four different possibilities of multipliers (coefficient of x^2), which are indicated below:

1. $Mn^2x^2 \pm k = y^2$ (multiplier is divisible by n^2)
2. $a^2x^2 \pm k = y^2$ (multiplier is a square)
3. $Nx^2 \pm N = y^2$ (multiplier and the additive are the same)
4. $-Nx^2 \pm k = y^2$ (multipler is negative)

3 Applications of algebra

3.1 Ekavarṇasamīkaraṇa and madhyamāharaṇa

The procedures outlined by Bhāskara for solving linear or quadratic, equations with a single variable – are referred to as *ekavarṇasamīkaraṇa* and *madhyamāharaṇa.*

In fact, the procedure for solving linear equations in one variable is very much similar to the techniques used in modern mathematics. In this section, by way of examples, Bhāskara covers many branches in mathematics, such as the rule of three (*trairāśika*), arithmetic progression (*śreḍīphala*) and geometry (*kṣetravyavahāra*).

Quadratic equations

From early times, Indians were aware of *madhyamāharaṇa* or quadratic equations. Bhāskara himself defines *madhyamāharaṇa* as [BīGa1927, p. 59]:

तच्च मध्यमाहरणमिति व्यावर्णयन्त्याचार्याः ।
यतोऽत्र वर्गराशौ एकस्य मध्यमस्याहरणमिति ॥

tacca madhyamāharaṇamiti vyāvarṇayantyācāryā: |
yato'tra vargarāśau ekasya madhyamasyāharaṇamiti ||

It is specifically described by the Ācāryas as *madhyamāharaṇa* because the middle term of the quadratic gets removed.

The most significant contribution came from Śrīdhara (ninth century) whose treatise is unfortunately lost to us. He gives an excellent method for solving the quadratic equation which is quoted by Bhāskara [BīGa1927, p. 61]:

चतुराहतवर्गसमरूपैः पक्षद्वयं गुणयेत् ।
पूर्वाव्यक्तस्य कृतेः समरूपाणि क्षिपेत्तयोरेव ॥

caturāhatavargasamarūpaiḥ pakṣadvayaṃ guṇayet |
pūrvāvyaktasya kṛteḥ samarūpāṇi kṣipettayoreva ||

Multiply both sides of the equation by a number equal to four times the [coefficient] of the square, and add to them a number equal to the square of the original [coefficient] of the unknown quantity. (Then extract the square root) [Col2005, p. 209].

This reading of the verse is found in the text of the commentator Kṛṣṇa Daivajña [BīPa1958, p. 188]. Colebrooke adds in his notes [Col2005, p. 210] that Rāmakṛṣṇa's text concurs with this reading. Datta and Singh [DA2001, p. 65] also accept the same. However Sūryadāsa in his *Sūryaprakāśa* [SūPr2001, f. 44], the other commentary of *Bījagaṇita*, gives a different reading for the second line as *avyakta-vargarūpaiḥ yuktau pakṣau tato mūlam.* According to Datta and Singh [DA2001, p. 65], Jñānarāja (sixteenth century CE) has also given the same reading in his *Bījagaṇita* and this has also been accepted by Sudhakara Dvivedi as is found in his edition [BīGa1927, p. 61].

Śrīdhara's rule

According to Śrīdhara, to find the roots of the equation $ax^2 + bx = c$, we need to multiply by $4a$ throughout. Thus we have,

$$4a^2x^2 + 4abx = 4ac.$$

Adding b^2 to both sides (in order to complete the square), and doing some algebraic manipulation, we get,

$$\begin{aligned} 2ax + b &= \sqrt{4ac + b^2} \\ \text{or} \qquad 2ax &= \sqrt{4ac + b^2} - b, \\ \text{and} \qquad x &= \frac{\sqrt{4ac + b^2} - b}{2a}. \end{aligned}$$

It is tacitly assumed that a, b, c are all positive.

Special features of the quadratic equation

While dealing with quadratic equations, a few examples are taken up by Bhāskara in order to illustrate:

(i) that every quadratic equation has two roots.

(ii) when there are two positive roots, one of them may not be suitable for a particular problem.

(iii) to explain situations when the negative root cannot be accepted.

Bhāskara gives a rule [BīGa1927, p. 59] to determine the two roots of a quadratic equation, laying emphasis on the words, *dvividham kvacittu* implying that **sometimes** two values of the unknown may be possible.

In the modern treatment, the fact that $4ac + b^2$ has two square roots (one positive and one negative), is indicated by attaching the symbol $\pm$ to $\sqrt{4ac+b^2}$. Bhāskara's rule states that $(2ax + b)^2 = \{-(2ax + b)\}^2$. Clearly, because of the concern for only practical problems to be solved, only those cases where $\sqrt{4ac+b^2} - b$ and $\sqrt{4ac+b^2} - b$ are both positive are considered. A necessary condition therefore would be $b < 0$ while $a, c > 0$. Bhāskara's rule says that b should be numerically greater than $\sqrt{4ac+b^2} - b$ for such a situation.

Bhāskara rules out negative roots though they can be allowed in numerical examples. However it should be remembered that Indian mathematicians used algebra to solve practical problems, which generally admit positive solutions and hence ignored negative solutions. The following example will make it clear [BīGa1927, p. 65]:

> यूथात् पञ्चांशकस्त्र्यूनो वर्गितो गह्वरं गतः ।
> दृष्टः शाखामृगः शाखां आरूढो वद ते कति ॥
>
> *yūthāt pañcāṃśakastryūno vargito gahvaraṃgataḥ |*
> *dṛṣṭaḥ śākhāmṛgaḥ śākhāṃ ārūḍho vada te kati ||*
>
> The fifth part of the troop of monkeys less three, squared, had gone to a cave and one monkey was in sight, having climbed on a branch. Say how many were there in all?

The two solutions are 50 and 5. In his commentary *vāsanā* Bhāskara adds [BīGa1927, p. 66]:

> द्वितीयमत्र न ग्राह्यमनुपपन्नत्वात्। न हि व्यक्ते ऋणगते लोकस्य प्रतीतिरस्ति।
>
> *dvitīyamatra na grāhyamanupapannatvāt | na hi vyakte ṛṇagate lokasya pratītirasti |*
>
> But the second in this case is not to be accepted; for it is incongruous. People do not conceive of negative numbers in manifest physical quantities [*vyakta*].

By chosing the following example, Bhāskara points out the error in the maxim given by Padmanābha:

> कर्णस्य त्रिलवेनोना द्वादशाङ्गुलशङ्कुभा ।
> चतुर्दशाङ्गुला जाता गणक ब्रूहि तां द्रुतम् ॥
>
> *karṇasya trilavenonā dvādaśāṅgulaśaṅkubhā |*
> *caturdaśāṅgulā jātā gaṇaka brūhi tāṃ drutam ||*
>
> The shadow of a gnomon 12 *aṅgulas* being lessened by a third part of the hypotenuse became 14 *aṅgulas* long. State quickly O mathematician! the length of the shadow.

The two solutions for this are $\frac{5}{2}$ and 9. Hence Bhāskara observes [BīGa1927, p. 67]:

> द्वितीयच्छाया चतुर्दशभ्यो न्यूनाऽतोऽनुपपन्नत्वात् न ग्राह्या। अत उक्तं द्विविधं क्वचिदिति।
>
> *dvitīyacchāyā caturdaśabhyo nyūnā'to'nupapannatvāt na grāhyā ata uktaṃ dvividhaṃ kvaciditi |*
>
> The second value of the shadow is less than 14; therefore by reason of its incongruity, it should not be taken. Hence it was said that this two-fold value holds in some cases.

Continuing his discussion further on the topic, Bhāskara cites Padmanābha and mentions that the rule stated by him is an exception to what has been given by Padmanābha [BīGa1927, p. 67]:

> अत्र पद्मनाभबीजे –
>
> > व्यक्तपक्षस्य चेन्मूलमन्यपक्षर्णरूपतः ।
> > अल्पं धनर्णगं कृत्वाद्विविधोत्पद्यते मितिः ॥
>
> इति यत् परिभाषितं तस्य व्यभिचारोऽयम्।
>
> *atra padmanābhabīje –*
>
> > *vyaktapakṣasya cenmūlamanyapakṣarṇarūpataḥ |*
> > *alpaṃ dhanarṇagaṃ kṛtvādvividhotpadyate mitiḥ ||*"

iti yat paribhāṣitaṃ tasya vyabhicāro'yam |

In this connection, it is said in the algebraic work of Padmanābha "When the root of the absolute side is less than the known number being negative on the other side, making it positive and negative, the value comes out two-fold" [Col2005, p. 218]. The rule given thus is only an exception of that.

Anekavarṇasamīkaraṇa and madhyamāharaṇa

As earlier, the term *varṇa* here refers to a variable or unknown quantity. Hence, the word *anekavarṇasamīkaraṇa* refers to the procedure for solving equations with more than one unknown. There are both simple and multiple equations of this type.

The *sūtras* explaining the method employed to solve the equations involving several unknowns are commented on by Bhāskara [BīGa1927, p. 76] himself in his auto commentary. Kṛṣṇa Daivajña observes [BīGa1927, p. 206]:

एतानि सूत्राणि आचार्यैरेव सम्यक् व्याख्यातानीति नास्माभिर्व्याक्रियन्ते ।

etāni sūtrāṇi ācāryaireva samyak vyākhyātānīti nāsmābhirvyākriyante |

These *sūtras* have already been well explained by the Ācārya and therefore not dealt with in detail by us.

A controversial example

An example by earlier authors is taken up by Bhāskara for discussion in this section [BīGa1927, p. 93]:

षडष्टशतकाः क्रीत्वा समार्घेण फलानि ये ।
विक्रीय च पुनः शेषं एकैकं पञ्चभिः पणैः ॥
जाताः समपणास्तेषां कः क्रयो विक्रयश्च कः ॥

ṣaḍaṣṭaśatakāḥ krītvā samārgheṇa phalāni ye |
vikrīya ca punaḥ śeṣamekaikaṃ pañcabhiḥ paṇaiḥ ||
jātāḥ samapaṇāsteṣāṃ kaḥ krayoḥ vikrayaśca kaḥ ||

Three traders having six, eight and a hundred, for their capitals, bought fruits at a uniform rate; and resold [a part] so; and disposed of the remainder at one for five *paṇas*; and thus became equally rich. What was [the rate of] their purchase? What was [the rate of] their sale? [Col2005, p. 242]

Bhāskara himself states that he has 'somehow' solved it. He has made his own views on the said problem quite clear in his commentary [BīGa1927, p. 99]:

इदं अनियताधारक्रियायां आद्यैरुदाहृत्य यथाकथञ्चित् समीकरणं कृत्वाऽऽनीतम्। इयं तथा कल्पना कृता यथा अत्र अनियताधारायामपि नियताधाराक्रियावत् फलं आगच्छति। एवंविध-कल्पनाच्च क्रियासङ्कोचात् यत्र व्यभिचरति तत्र बुद्धिमद्भिः बुद्ध्या सन्धेयम्।

idaṃ aniyatadhārakriyāyāṃ ādyairudāhṛtya yathākathañcit samīkaraṇaṃ kṛtvā''nītam | iyaṃ tathā kalpanā kṛtā yathā atra aniyatadhārākriyāvat phalaṃ āgacchati | evaṃ vidhakalpanācca kriyāsaṅkocāt yatra vyabhicarati tatra buddhimadbhiḥ buddhyā sandheyam ||

This (example) was told by our predecessors for the case of insecure computation and solved after an equation was somehow made. Here, an assumption was made (by me) in such a way that the computation, through its grounds were insecure, might come to an answer like a computation with secure grounds. Wherever (a compution) is deviating from the right course due to the contraction of computaion based on this kind of assumption, (the answer) should be reconciled (with the statements) by means of intelligence by intelligent people [BīGa2009, p. 152].

The above example cited by Bhāskara raises two important issues:

(a) Whether a *trairāśika* should be used or not, and

(b) Whether a given fraction should be reduced to smaller numbers before doing the *kuṭṭaka.*

In an elaborate discussion Kṛṣṇa Daivajña defends Bhāskara's use of the *trairāśika* in this example [BīPa1958, pp. 225–232]. Kṛṣṇa justifies the second point by quoting Bhāskara's own words from *Golapraśnādhyāya* [SiŚi1929, v. 24]:

उद्दिष्टं कुट्टके तज्ज्ञैः ज्ञेयं निरपवर्तनम् ।
व्यभिचारः क्वचित् क्वापि खिलत्वापत्तिरन्यथा ॥

uddiṣṭaṃ kuṭṭake tajjñaiḥ jñeyaṃ nirapavartanam |
vyabhicāraḥ kvacit kvāpi khilatvāpattiranyathā ||

It has been established by the scholars that reduction should be done in a *kuṭṭaka*; [but] 'no reduction' should be done if it will lead to an incorrect answer.

Kamalākara (seventeenth century) does not agree with Kṛṣṇa's view. In his *Siddhāntatattvaviveka* [STV1991, v. 255], he says:

नवाङ्कुरेऽपि बीजोत्थे कुट्टकानपवर्तने ।
सिद्धान्तसंमतिर्योक्ता सदर्थाज्ञानतोऽस्ति सा ॥

navāṅkure'pi bījotthe kuṭṭakānapavartane |
siddhāntasaṃmatiryoktā sadarthājñānato'sti sā ||

The statement, even in the *Navāṅkura* (of Kṛṣṇa), that the said *anapavartana* has sanction from *Siddhānta* (*śiromaṇi*) is due to a lack of proper understanding.

Sudhakara Dvivedi in his edition of the *Bījagaṇita* takes objection to the method propounded by Bhāskara. He says [BīGa1927, p. 94]: "Bhāskara has

given the *sūtra* on *kuṭṭaka* stressing the need for *apavartana.* To say that here it is not applicable is only satisfying the duller intellects." In this connection, he also adds in a different context: "While explaining the example *ṣaḍṣṭaśatakaḥ*, whatever Kṛṣṇa has said without understanding the real import of Bhāskara's statement given in the *Golapraśnādhyāya,* is to be looked into by mathematicians" [GaTa1892, p. 181].

In the same edition of Dvivedi's *Bījagaṇita,* in his expository notes, Muralidhara Jha does not concur with Dvivedi. He gives the following rejoinder [BīGa1927, p. 94]: "Bhāskara has not done *apavartana* knowing fully that by doing so a solution will not be possible (in this context). But he has also added that in certain exceptional cases, such assumptions should be made by the wise. So it is not correct to use the phrase *mandānandakarī* (pleasing the dull-witted)". Thus the above example has prompted a lot of discussion even up to recent times, which indeed vouches for the popularity of the *Bījagaṇita.*

Madhyamāharaṇa

Bhāskara is perhaps the earliest author to mention the solution to indeterminate equations of the second degree other than *vargaprakṛti* in his *Bījagaṇita.* But since he himself has taken examples from other authors it is highly probable that the method was known to mathematicians before him. However, Datta and Singh [DA2001, p. 181] observe: "Neither those illustrations nor a treatment of equations of these types occur in the algebra of Brahmagupta or in any other work anterior to Bhāskara II." According to Bhāskara there are two types of these equations. He adds [BīGa1927, p. 106]:

> एवं तदैव यदा असकृत् समीकरणं यदा तु सकृदेव समीकरणं तदा एकं वर्णं मुक्त्वाऽन्येषामिष्टानि मानानि कृत्वा प्राग्वन्मूले।
>
> *evaṃ tadaiva yadā asakṛt samīkaraṇaṃ yadā tu sakṛdeva samīkaraṇaṃ tadā ekaṃ varṇaṃ muktvā'nyeṣāmiṣṭāni mānāni kṛtvā prāgvanmūle* |
>
> This is to be practised when there is more than one equation (*asakṛt samīkaraṇa*). But if there be only one (*sakṛt samīkaraṇa*), then reserving a single colour, and putting arbitrary values for the rest, let the root be sought as before [Col2005, p. 252].

Simple equation

Considering equations of the type $ax^2 + bx + c = y^2$, the example given by Bhāskara is [BīGa1927, p. 101]:

को राशिर्द्विगुणो राशिवर्गैः षड्भिः समन्वितः ।
मूलदो जायते बीजगणितज्ञ वदाऽऽशु तम् ॥

ko rāśirdviguṇo rāśivargaiḥ ṣaḍbhiḥ samanvitaḥ |
mūlado jāyate bījagaṇitajña vadā"śu tam ||

What number when doubled and added to six times its square becomes capable of yielding a square root? O algebraist, tell me quickly.

The problem posed in the above verse may be written as,

$$6x^2 + 2x = y^2.$$

Multiplying by 6 throughout, we get

$$36x^2 + 12x = 6y^2.$$

Adding 1 to both sides, to complete the square in the LHS we have

$$(6x + 1)^2 = 6y^2 + 1.$$

Denoting $6x + 1$ by X, we obtain the standard *vargaprakṛti* equation

$$X^2 = 6y^2 + 1,$$

whose solution sets are (2, 5), (20, 49) and so on. For example when $X = 5, y = 2$, we get $x = \frac{2}{3}$.

Using the above method of Bhāskara, an infinite number of solutions can be obtained. This easy method has also been followed by later authors like Nārāyaṇa (ca.1350 CE) and Jñānarāja according to Datta and Singh. They add that some of these methods were rediscovered in 1733 by Euler and that: "His (Euler's) method is indirect and cumbrous. Lagrange's (eighteenth century CE) begins in the same way as that of Bhāskara by completing the square on the left side of the equation" [DA2001, p. 186].

Vargakuṭṭaka

The indeterminate equation of the type $x^2 = by + c$ is called *vargakuṭṭaka*. It derives its name from the fact that it closely resembles the linear *kuṭṭaka* equation in as much as the problem reduces to finding a square which when reduced by c will be exactly divisible by b. Earlier authors like Brahmagupta [BSS1966, vol. 4, p. 79] adopted the method of assuming suitable arbitrary values for y and then solving for x. However neither Brahmagupta nor his

commentator, Pṛthūdakasvāmī have given any general solution. Bhāskara was the first to give a general solution [BīGa1927, p. 120]:

वर्गादेर्यो हरस्तेन गुणितं यदि जायते ।
अव्यक्तं तत्र तन्मानमभिन्नं स्याद्यथा तथा ।
कल्प्योऽन्यवर्णवर्गादिः तुल्यं शेषं यथोक्तवत् ॥

vargāderyo harastena guṇitaṃ yadi jāyate |
avyaktaṃ tatra tanmānamabhinnaṃ syādyathā tathā |
kalpyo'nyavarṇavargādiḥ stulyaṃ śeṣaṃ yathoktavat ||

If the simple unknown be multiplied by the quantity which was divisor of the square etc., [on the other side]; then, that its value may in such case be an integer, a square or like [term] of another symbol must be put equal to it; the rest [of the operations] will be as before taught [Col2005, p. 262].

Without getting into further details we simply mention that– here three cases are considered by Bhāskara:

(a) If c is a square, then y should be assumed as $bz^2 + 2\sqrt{cz}$ so that $by + c$ is a square, where z is an arbitrary number.
(b) When c is not a square, assume x to be $\sqrt{by} + \sqrt{c}$.
(c) When the equation is of the form $ax^2 = by \pm c$, multiply by a to make it a *varga kuṭṭaka*.

3.2 Bhāvita

Bhāvita as a mathematical term refers not only to the product of two unknown variables such as xy, ab and so on, but also to the genre of equations dealing with products. Brahmagupta [BSS1966, ch. 18.62–63] refers to an equation of this type in his work. Similar equations occur in the *Bakhṣālī* Manuscript [DA2001, p. 297] and in the *Siddhāntaśekhara* of Śrīpati [DA2001, p. 302].

Bhāskara's aim was to give integral solutions while solving the *Bhāvita* equation. He himself states that such equations can be solved both using figures and algebraically [BīGa1927, *Bhāvita*, p. 125]:

सा च द्विधा सर्वत्र स्यादेका क्षेत्रगताऽन्या राशिगतेति।

sā ca dvidhā sarvatra syādekā kṣetragatā'nyā rāśigateti |

That is of two kinds everywhere, one using figures and the other using numbers.

Just to give a feel for the kind of equation that *bhāvita* refers to, we may cite an example [BīGa1927, p. 125]:

चतुस्त्रिगुणयो राश्योः संयुतिर्द्वियुता तयोः ।
राशिघातेन तुल्या स्यात् तौ राशी वेत्सि चेद्वद ॥

catustriguṇayo raśyoḥ saṃyutirdviyutā tayoḥ |
rāśighātena tulyā syāt tau raśī vetsi cedvada ||

Tell me if thou know two numbers such, that the sum of them, multiplied separately by four and by three, may, when added to two, be equal to the product of the same numbers.

If x and y are two unknowns to be determined, then the problem given above may be expressed using algebraic notation as follows.

$$xy = 4x + 3y + 2.$$

The solution to this problem can be obtained either geometrically or algebraically. In what follows we demonstrate this using both the approaches as outlined in the *Bījapallava* [BīGa1927, pp. 125–127].

Geometric demonstration

The product xy can be visualized as the area of a rectangle with sides x and y. It is evident from Figure 1, that the area of the rectangle $ABCD$ can be conceived to be made up of four other rectangles. That is,

$$ABCD = ASRP + SBTR + PRQD + RTCQ.$$

Taking the interstice between two lines to represent one unit, the area of the rectangle $ABTP = 4x$. Similarly the area of the rectangle $SBCQ = 3y$.

When the two rectangles are added, the rectangle $SBTR$ has been added twice, and so one has to be discarded. From the figure we have:

$$xy = 4x + 3y - 12 + \text{rectangle } PRQD.$$

It is given that $xy = 4x + 3y + 2$. That is,

$$4x + 3y - 12 + \text{rectangle } PRQD = 4x + 3y + 2.$$

This implies that the area of the rectangle $PRQD = 2 + 12 = 14$. This being the product of two adjacent sides p and q, the sides can be (1, 14) or (2, 7). If we take $p = 1, q = 14$, then $x = 5, y = 17$.

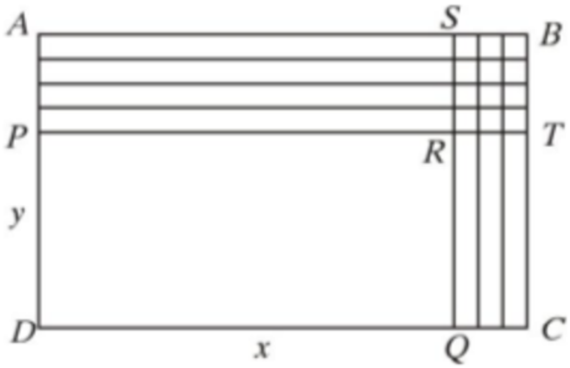

Figure 1: Finding the area of a rectangle.

Algebraic demonstration

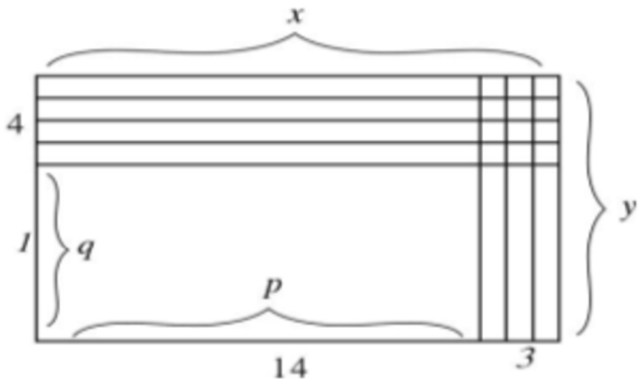

Figure 2: Geometric demonstration of algebraically finding the area of a rectangle.

Considering the same figure shown in the previous section, let $x = p+3,\ y = q+4$. Substituting for x and y in the given equation,

$$(p+3)(q+4) = 4(p+3) + 3(q+4) + 2.$$

Solving the above equation we obtain $pq = 14$. The rest of working is as before.

4 Conclusion

Bhāskarāchārya was one of the earliest of ancient Indian mathematicians to have treated algebra as a separate genre of mathematics. Also, it was Bhāskara, who for the time elaborately dealt with both the tools of algebra and its application. India reached the pinnacle of its achievement when Bhāskara explained the *cakravāla* method to solve the *vargaprakṛti* equation $Nx^2 + 1 = y^2$ many centuries ahead of the Europeans in his text *Bījagaṇita.*

We would like to conclude the article with the last verse of *Bījagaṇita* [BīGa1927, p. 130] that aptly sums up his text:

गणकभणितरम्यं बाललीलावगम्यं
सकलगणितसारं सोपपत्तिप्रकारम् ।
इति बहुगुणयुक्तं सर्वदोषैर्विमुक्तं
पठ पठ मतिवृद्ध्यै लघ्विदं प्रौढसिद्ध्यै ॥

gaṇakabhaṇitaramyaṃ bālalīlāvagamyaṃ
sakalagaṇitasāraṃ sopapattiprakāram |
iti bahuguṇayuktaṃ sarvadoṣairvimuktaṃ
paṭha paṭha mativṛddhyai laghvidaṃ prauḍhasiddhyai ||

To augment wisdom and strengthen confidence, read, do read, mathematician, this abridgment elegant in style, easily understood by youth, comprising the whole essence of computation, and containing the demonstration of its principles, replete with excellence and void of defect [Col2005].

A critical study of algorithms in the *Karaṇīṣaḍvidham*

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1 Introduction

Bhāskara's *Bījagaṇitam* (*BB*) contains 11 chapters consisting of 187 verses pertaining to algebra, 2 introductory verses and 9 verses in the epilogue [BīGa1980, pp. 7–59]. In the second verse of the epilogue Bhāskara admits that this text is compiled by selecting notable ideas from his predecessors like Brahmagupta, Śrīdhara, Padmanābha and others [BīGa1980, p. 52]. This is certainly the case in the chapter entitled *karaṇīṣaḍvidham* (six operations on the surds).

Prior to Bhāskara, we find the frequent use of terms like *dvi-karaṇī*, *tri-karaṇī*, *tṛtīya-karaṇī* etc., and the use of elementary operations like addition, multiplication and rationalization on *karaṇīs* in the *Śulvasūtras* [Dat1991, pp. 187–206]. Later Brahmagupta in the *Brāhmasphuṭasiddhānta* (628 CE) discussed six operations (addition, subtraction, multiplication, division, square and square roots) on *karaṇīs* in three verses [BSS1966, vv. 38–40, pp. 1198–1204]. Some operative rules on *karaṇīs* are discussed by Bhāskara I in his *Āryabhaṭīya-bhāṣya* (c. 629 CE) [AB1976, pp. xxi–xxii].

Mahāvīra (850 CE) in the *Gaṇitasārasaṅgraha* stated the rule for addition and subtraction of *karaṇīs* in a single verse. [GSS2000, v. 88, p. 479] and Śrīpati in the *Avyaktagaṇitādhyāya* of the *Siddhāntaśekhara* (1039 CE) discussed the same six operations in five verses [Sin1986, vv. 7–12, p. 29]. However the works of Padmanābha (c. 700 CE) and Śrīdhara (750 CE) on algebra are not available. Later Nārāyaṇa Paṇḍita (c. 1356 CE) in the *Bījagaṇitāvataṃsa* (BGA) followed Bhāskara with some traditional algorithms.

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Bhāskara elaborated on the same six operations in the fourth chapter, with the help of 13 *sūtras/vidhis* (formulae/algorithms) supported by 17 *uddeśakas* (examples) spread over a total of 22 verses [BīGa1980, pp. 13–17]. The meaning of the title is explained by Kṛṣṇa Daivajña in his Sanskrit commentary the *Navāṅkura* (*NK*) on *BB* as [BīGa1930, p. 35]:

...तत्र यस्य राशेर्मूलेऽपेक्षिते निरग्रं मूलं न संभवति स करणी ।

..tatra yasya rāśermūle'pekṣite niragraṃ mūlaṃ na saṃbhavati sa karaṇī |

That is, that quantity of whose square root is expected (but) not possible without a remainder is *karaṇī*, and,

...मूलराश्योर्वर्गद्वारा यत्षड्विधं तत्करणीषड्विधमिति ।

...mūlarāśyorvargadvārā yatṣaḍvidhaṃ tatkaraṇīṣaḍvidhamiti |

That is, six operations performed with squares of two square root quantities are [termed as] *karaṇīṣaḍvidham*. For a detailed discussion on the concept of the related operations refer to [Dat1993] and [Pat2005].

This paper focuses on these 22 verses. Here, a simple translation of the verses is provided and algorithms therein are explained with comments. The sequence of the verses in the original text or part of them is sometimes changed to facilitate the development of the subject. The examples stated by Bhāskara are noted and some are solved. This author has also introduced five new illustrations. The technical terms and the word numerals used in this article along their mathematical meaning, are listed below in Table 1 and 2.

2 The sum and the difference of two *karaṇīs*

Bhāskara states two methods for the sum and difference of two surds followed by an existence condition and three examples.

First method

Bhāskara says [BīGa1980, p. 13]:

योगं करण्योर्महतीं प्रकल्प्य घातस्य मूलं द्विगुणं लघुं च ।
योगान्तरे रूपवदेतयोः...

yogaṃ karaṇyormahatīṃ prakalpya ghātasya mūlaṃ dviguṇaṃ laghuṃ ca |
yogāntare rūpavadetayoḥ...

Table 1: Technical terms and their meanings.

Sanskrit terms	Mathematical meaning	Sanskrit terms	Mathematical meaning	Sanskrit terms	Mathematical meaning
आदि	first term	वर्ग, कृति	square (term/operation)	वर्गराशि	square expression
खण्ड	part	महती	greater term	विधि	algorithm
गुणक	multiplier	मूल, पद	square root	शेष	remainder
गुण्य	multiplicand	योग, सङ्कलित	sum	स्वहत	multiplied by itself
गुणयेत्	multiply	योगान्तर	sum and difference	हत	multiplied
घात, वध	product	योगज	obtained by addition	हर	divisor
छेद	denominator	राशि	quantity/number	धन	positive
रूप	integer	लघु	smaller term	हृत	divided
अन्तर	difference	विश्लेष	splitting	भजेत	divide

Table 2: Words and numbers represented by words.

Object numeral	Number	Object numeral	Number	Object numeral	Number	Object numeral	Number
हुताशन	3	रुद्र	11	तिथि	15	दन्त	32
ऋतु	6	अर्क, सूर्य	12	सिद्ध	24		
नाग, गज	8	विश्व	13	भ	27		

For the sum or difference of the two surds, first treat them as integers. Recognize the sum [of the integers] as *mahatī* [the greater quantity] and twice the square root of the product [of the same integers] as *laghu* [the smaller quantity]. [The required sum or difference is the square root of the sum or difference of *mahatī* and *laghu* respectively].

Algorithm:

For $\sqrt{x}$ and $\sqrt{y}, x > y$, we find $(x + y)$ and $2\sqrt{xy}$,

Then, $\sqrt{x} \pm \sqrt{y} = \pm\sqrt{(x + y) \pm 2\sqrt{xy}}$.

Explanation: $\sqrt{x} \pm \sqrt{y} = \pm\sqrt{\left(\sqrt{x} \pm \sqrt{y}\right)^2} = \pm\sqrt{x + y \pm 2\sqrt{xy}}$.

Second method

Bhāskara says [BīGa1980, p. 13]:

लघ्व्या हृतायास्तु पदं महत्याः सैकं निरेकं स्वहतं लघुघ्नम् ।
योगान्तरे स्तः क्रमशः तयोः ...

laghvyā hṛtāyāstu padaṃ mahatyāḥ
saikaṃ nirekaṃ svahataṃ laghughnam |
yogāntare staḥ kramaśaḥ tayoḥ ...

Divide the greater number by the smaller one and find the square root. [Separately] add 1 and subtract 1 [to the quotient] and square. Multiply each term by the smaller number. [The required sum or difference is the square root of sum or difference of obtained results respectively].

Algorithm:

For $\sqrt{x}$ and $\sqrt{y}, x > y$, we find $\sqrt{\frac{x}{y}}, \left(\sqrt{\frac{x}{y}} + 1\right)^2$ and $\left(\sqrt{\frac{x}{y}} - 1\right)^2$.

Then, $\sqrt{x} \pm \sqrt{y} = \pm\sqrt{y \times \left(\sqrt{\frac{x}{y}} \pm 1\right)^2}$.

Explanation: $\sqrt{x} \pm \sqrt{y} = \sqrt{y}\left(\sqrt{\frac{x}{y}} \pm 1\right) = \pm\sqrt{y \times \left(\sqrt{\frac{x}{y}} \pm 1\right)^2}$.

Existence conditions and examples

Bhāskara states [BīGa1980, p. 13]:

पृथक् स्थितिः स्याद्यदि नास्ति मूलम् ।

pṛthaka sthitiḥ syād yadi nāsti mūlam |

If the square root does not exist then the sum or difference will remain separated.

If the square root of xy or $\frac{x}{y}$ does not exist, then $\left(\sqrt{x} \pm \sqrt{y}\right)$ remains as it is.

Examples [1–3]: Bhāskara gives the following three examples [BīGa1980, p. 14]:

द्विकाष्टमित्योस्त्रिभसङ्ख्ययोश्च योगान्तरे ब्रूहि पृथक् करण्योः ।
त्रिसप्तमित्योश्च ...

dvikāṣṭamityostribhasaṅkhyayośca yogāntare brūhi pṛthak karaṇyoḥ |
trisaptamityośca ...

Tell separately the sum and difference of the *karaṇīs* 2 and 8; 3 and 27; and 3 and 7.

Essentially the problem is to find: [1]: $\sqrt{2} \pm \sqrt{8}$. [2]: $\sqrt{3} \pm \sqrt{27}$. [3]: $\sqrt{3} \pm \sqrt{7}$.

Comments:

1. For any two positive rational numbers x and y such that $x > y$, we have $\sqrt{x} - \sqrt{y} > 0$. Squaring we get, $x + y - 2\sqrt{xy} > 0$. That is, $x + y > 2\sqrt{xy}$. Hence the terms *mahatī* and *laghu* are justified. This guarantees that $(x + y \pm 2\sqrt{xy})$ is always positive and hence, its square root exists. Again, we have $y \times \left(\sqrt{\frac{x}{y}} \pm 1\right)^2$ is positive for all x and y hence, its square root exists.
2. In the third example, the product 21 is not a perfect square number hence, the sum does not exist.
3. This amounts to saying that only like surds can be added or subtracted.

Illustration 1: For $x = \dfrac{6}{7}, y = \dfrac{8}{21}$, find $\sqrt{x} \pm \sqrt{y}$.

Here $xy = \dfrac{16}{49}$, a perfect square rational number. Now,

$$x + y = \frac{26}{21}, \quad 2\sqrt{xy} = \frac{8}{7}.$$

Hence,

$$\sqrt{x} \pm \sqrt{y} = \sqrt{\frac{26 \pm 24}{21}}.$$

Thus,

$$\sqrt{x} + \sqrt{y} = \sqrt{\frac{50}{21}}, \quad \sqrt{x} - \sqrt{y} = \sqrt{\frac{2}{21}}.$$

3 *Viśleṣa-sūtra* (Rule for splitting)

Bhāskara is the first scholar to state the algorithm under the heading *viśleṣa-sūtra*, which is the rule for splitting. Here, we express the given *karaṇī* as the sum or difference of two or more *karaṇīs*. Before we present the rule, we explain the concept of *yogaja-karaṇī*.

A *karaṇī* obtained by the addition of two or more *karaṇīs* is called *yogaja-karaṇī*. For example, $\sqrt{3}+\sqrt{12}+\sqrt{147} = \sqrt{300}$. The criterion for locating this *karaṇī* is that, $\sqrt{x}$ is a *yogaja-karaṇī* if x is divisible by m^2, for some integer m. For example, $\sqrt{32}, \sqrt{245}$ are *yogaja-karaṇīs* but $\sqrt{17}$ is not.

Bhāskara states the algorithm for the splitting of this *yogaja-karaṇī* as follows [BīGa1980, p. 15]:

वर्गेण योगकरणी विहृता विशुद्ध्येत् खण्डानि तत्कृतिपदस्य यथेप्सितानि ।
कृत्वा तदीयकृतयः खलु पूर्वलब्ध्या क्षुण्णा भवन्ति पृथगेवमिमाः करण्यः ॥

vargeṇa yogakaraṇī vihṛtā viśuddhyet
khaṇḍāni tatkṛtipadasya yathepsitāni |
kṛtvā tadīyakṛtayaḥ khalu pūrvalabdhyā
kṣuṇṇā bhavanti pṛthagevamimāḥ karaṇyaḥ ||

Select the square number by which the given *yogaja-karaṇī* is exactly divisible and find the quotient. By breaking up the square root of this square into [rational] parts at will, multiply the square of each part by the previous quotient. [Then], *yogaja-karaṇī* splits into [the sum or difference of] these different *karaṇīs*.

Algorithm:

For $\sqrt{N}$, let m^2 exactly divide N. Then find $\frac{N}{m^2} = y$ (say). Select rational numbers m_1, m_2, m_3, etc. such that $m = m_1 \pm m_2 \pm m_3 \pm$ ... and find $y \times m_1^2, y \times m_2^2, y \times m_3^2$,...Then,

$$\sqrt{N} = \sqrt{ym_1^2} \pm \sqrt{ym_2^2} \pm \sqrt{ym_3^2} \pm \ldots .$$

Explanation:

For $N = m^2 y$, we have, $\sqrt{N} = m\sqrt{y}$. If we express $m = m_1 \pm m_2 \pm m_3$...,

then, $\sqrt{N} = (m_1 \pm m_2 \pm m_3 \ldots)\sqrt{y} = \sqrt{ym_1^2} \pm \sqrt{ym_2^2} \pm \sqrt{ym_3^2} \pm$...

The positive or negative sign is selected according to the sign of the corresponding part.

Illustration 2:

Split $\sqrt{28}$ in two different ways using *viśleṣa-sūtra*.

Here, $N = 28$. Select $m^2 = 4$, then $y = 7$.

If $m = 2 = 5 + 6 + 3 - 12$ then, $\sqrt{28} = \sqrt{175} + \sqrt{252} + \sqrt{63} - \sqrt{1008}$.

If $m = 2 = \dfrac{1}{3} + \dfrac{5}{3}$ then, $\sqrt{28} = \sqrt{\dfrac{7}{9}} + \sqrt{\dfrac{175}{9}}$.

4 Multiplication of the *karaṇīs*

Bhāskara first states the method for the multiplication of an integer and a surd, and then refers a method for the multiplication of surd expressions followed by two examples.

Multiplication and division of the *karaṇī* by an integer

Bhāskara says [BīGa1980, p. 13]:

> वर्गेण वर्गं गुणयेद्भजेच्च ।
>
> *vargeṇa vargaṃ guṇayedbhajecca* |
>
> That is, multiply or divide the square of [the *karaṇī*] by the square [of the multiplier].

Kṛṣṇa Daivajña explains this as [BīGa1930, p. 36]:

> करणीगुणने कर्तव्ये यदि रूपाणां गुण्यत्वं गुणकत्वं वा स्यात् करणीभजने वा कर्तव्ये यदि रूपाणां भाज्यत्वं वा भाजकत्वं वा स्यात् तदा रूपाणां वर्गं कृत्वा गुणनभजने कार्ये।
>
> *karaṇīguṇane kartavye yadi rūpāṇāṃ guṇyatvaṃ guṇakatvaṃ vā syāt karaṇībhajane vā kartavye yadi rūpāṇāṃ bhājyatvaṃ vā bhājakatvaṃ vā syāt tadā rūpāṇāṃ vargaṃ kṛtvā guṇanabhajane kārye* |
>
> That is, if the surds need to be multiplied by or divided by an integer, or if an integer is to be multiplied by or divided by the surd, then perform it by squaring the integer.

Algorithm:

For any integer m, we have

$$m \times \sqrt{x} = \sqrt{m^2 x}, \quad \frac{\sqrt{x}}{m} = \sqrt{\frac{x}{m^2}} \quad \text{and} \quad \frac{m}{\sqrt{x}} = \sqrt{\frac{m^2}{x}}.$$

Comment:

In general, $\sqrt{m} \times \sqrt{x} = \sqrt{mx}$ and $\dfrac{\sqrt{x}}{\sqrt{m}} = \sqrt{\dfrac{x}{m}}$.

Multiplication of the surd expressions

In the previous chapter of *BB* titled '*Varṇaṣaḍvidham*' (Six operations on algebraic numbers) Bhāskara says [BīGa1980, p. 11]:

अव्यक्तवर्गकरणीगुणनासु चिन्त्यो व्यक्तोक्तखण्डगुणनं विधिरेवमत्र ।
avyaktavargakaraṇīguṇanāsu cintyo vyaktoktakhaṇḍaguṇanaṃ vidhirevamatra |

That is, the *khaṇḍaguṇana* method [for the multiplication of numbers] stated in *vyakta* (*gaṇita*) (i.e., in the *Līlāvatī* [Pha1971, ch. 4.15, p. 8]) should be considered here for the square of the polynomial unknowns and for the multiplication of the polynomial *karaṇīs* [surd expressions].

Comment:

The process of *Khaṇḍaguṇana* is akin to the distributive property of multiplication over the addition of numbers, variables or surds.

Examples [4–5]: To illustrate this, Bhāskara gives the following two examples [BīGa1980, p. 14]:

द्वित्र्यष्टसंख्यागुणकः करण्योः गुण्यस्त्रिसंख्या च सपञ्चरूपा ।
वधं प्रचक्ष्वाऽऽशु विपञ्चरूपे गुणेऽथवा त्र्यर्कमिते करण्यौ ॥
dvitryaṣṭasaṃkhyāguṇakaḥ karaṇyoḥ guṇyastrisaṃkhyā ca sapañcarūpā |
vadhaṃ pracakṣvā"śu vipañcarūpe guṇe'thavā tryarkamite karaṇyau ||

Quickly find the product, if the multiplicand is *karaṇī* 3 with integer 5 and the multiplier [consists of the addition of] *karaṇīs* 2, 3 and 8 or *karaṇīs* 3 and 12 and integer (-5).

The problem given here is to find:

$$[4]\text{: } (\sqrt{3}+5)(\sqrt{2}+\sqrt{3}+\sqrt{8}), \qquad [5]\text{: } (\sqrt{3}+5)(\sqrt{3}+\sqrt{12}-5).$$

5 Division of the *karaṇīs*

Bhāskara says [BīGa1980, p. 14]:

धनर्णताव्यत्ययमीप्सितायाः छेदे करण्या असकृद्विधाय ।
तादृक्छिदा भाज्यहरौ निहन्यात् एकैव यावत् करणी हरे स्यात् ॥
भाज्यास्तया भाज्यगताः करण्यः लब्धाः करण्यो यदि योगजाः स्युः ।
विश्लेषसूत्रेण पृथक् च कार्याः तथा यथा प्रष्टुरभीप्सिताः स्युः ॥
dhanarṇatāvyatyayamīpsitāyāḥ chede karaṇyā asakṛdvidhāya |
tādṛkchidā bhājyaharau nihanyāt ekaiva yāvat karaṇī hare syāt ||

bhājyāstayā bhājyagatāḥ karaṇyaḥ labdhāḥ karaṇyo yadi yogajāḥ syuḥ |
viśleṣasūtreṇa pṛthak ca kāryāḥ tathā yathā praṣṭurabhīpsitāḥ syuḥ ||

Change the sign of the one optionally chosen *karaṇī* of the divisor. Multiply by this new *karaṇī*, the dividend and the [original] divisor. Repeat such procedure till only one *karaṇī* remains in the divisor. Divide the *karaṇīs* in the dividend by that single *karaṇī*. If any of the *karaṇīs* thus obtained is a *yogaja-karaṇī* then split it by the *viśleṣa-sūtra* as per the wishes of the interrogator.

Comment:

This algorithm is same as the rationalization method except for the last two steps.

6 Square of the sum of the *karaṇīs*

Bhāskara does not specify any algorithm here, because he has already described the *khaṇḍaguṇana* method for the multiplication (hence for the square) of polynomial surd. Here, he observes that [BīGa1980, p. 16]:

एकादिसङ्कलितमितकरणीखण्डानि वर्गराशौ स्युः ।

ekādisaṅkalitamitakaraṇīkhaṇḍāni vargarāśau syuḥ |

That is, in a square expansion, the [total] number of terms will be the sum from first term 1 [up to the number of *karaṇīs* involved in squaring].

Further, in an explanatory note he says [BīGa2008, p. 53]:

करणीवर्गराशौ रूपैरवश्यं भवितव्यम्। एककरण्या वर्गे रूपाण्येव, द्वयोः सरुपैका करणी, तिसृणां तिस्रः, चतसृणां षट्, पञ्चानां दश, षण्णां पञ्चदश इत्यादि।

karaṇīvargarāśau rūpairavaśyaṃ bhavitavyam ekakaraṇyā varge rūpāṇyeva, dvayoḥ sarupaikā karaṇī, tisṛṇāṃ tisraḥ, catasṛṇāṃ ṣaṭ, pañcānāṃ daśa, ṣaṇṇāṃ pañcadaśa ityādi |

That is, in the square of an expression having [all] the surd terms, certainly there will be a rational term. In the square of a single surd [there will be] only one rational term. [in the square of] two surds, one rational term along with one surd; of three surds, three irrational terms; of four surds, six irrational terms; of five surds, ten and of six surds fifteen [irrational terms] etc.

Comment:

The expansion $(a_1 \pm a_2 \pm a_3 \pm \ldots \pm a_n)^2$ will, obviously, have $\dfrac{n(n+1)}{2}$ terms. But in the expansion $(\sqrt{a_1} \pm \sqrt{a_2} \pm \sqrt{a_3} \pm \ldots \pm \sqrt{a_n})^2$, each term of the type $(\sqrt{a_i})^2, i = 1$ to n becomes the rational number and when all these n numbers are added, the result is a rational number. Thus, the later expansion

has one rational number and the remaining $\dfrac{n(n+1)}{2} - n = \dfrac{(n-1)n}{2}$ are the irrational terms. This is illustrated by Bhāskara for $n = 2, 3, 4, 5$ and 6.

Algorithm:

$$(\sqrt{a_1} \pm \sqrt{a_2} \pm \ldots \pm \sqrt{a_n})^2 = \sum a_i \pm \sum \sqrt{4a_ia_j}, \quad i < j.\ for\ i, j = 1 \text{ to } n.$$

Illustration 3:

$$\begin{aligned}(\sqrt{3} - \sqrt{7} + \sqrt{5})^2 &= (3 + 7 + 5) - \sqrt{4.3.7} + \sqrt{4.3.5} - \sqrt{4.7.5} \\ &= 15 - \sqrt{84} + \sqrt{60} - \sqrt{140}.\end{aligned}$$

Examples [6–11]: Bhāskara states the following six examples [BīGa1980, p. 15] and [BīGa1980, p. 16].

द्विकत्रिपञ्चप्रमिताः करण्यः तासां कृतिं द्वित्रिकसंख्ययोश्च ।
षट्पञ्चकद्वित्रिकसंमितानां पृथक् पृथङ्मे कथयाऽऽशु विद्वन् ॥
अष्टादशाष्टद्विकसंमितानां कृती कृतीनां च सखे पदानि ॥
त्रिसप्तमित्योर्वद मे करण्योः विश्लेषवर्गं कृतितः पदं च ।
द्विकत्रिपञ्चप्रमिताः करण्यः स्वस्वर्णगा व्यस्तधनर्णगा वा ॥
तासां कृतिं ब्रूहि कृतेः पदं च चेत् षडविधं वेत्सि सखे करण्याः ॥

dvikatripañcapramitāḥ karaṇyaḥ tāsāṃ kṛtiṃ dvitrikasaṃkhyayośca |
ṣaṭpañcakadvitrikasaṃmitānāṃ pṛthak pṛthaṅme kathayā”śu vidvan ||
aṣṭādaśāṣṭadvikasaṃmitānāṃ kṛtī kṛtīnāṃ ca sakhe padāni ||
trisaptamityorvada me karanyorviśleṣavargaṃ kṛtitaḥ padaṃ ca |
dvikatripañcapramitāḥ karaṇyaḥ svasvarṇagā vyastadhanarṇagā vā ||
tāsāṃ kṛtiṃ brūhi kṛteḥ padaṃ ca cet ṣaḍavidhaṃ vetsi sakhe karaṇyāḥ ||

Find separately the squares of

[6] : $\sqrt{2} + \sqrt{3} + \sqrt{5}$
[7] : $\sqrt{2} + \sqrt{3}$
[8] : $\sqrt{6} + \sqrt{5} + \sqrt{2} + \sqrt{3}$
[9] : $\sqrt{18} + \sqrt{8} + \sqrt{2}$
[10] : $\sqrt{3} + \sqrt{7}$
[11] : $\sqrt{2} + \sqrt{3} - \sqrt{5}$

In the later part of verse 46, Bhāskara also asks the reader to find the square of the surd expression by changing the signs in the last two examples and to determine the square root of the obtained square.

7 Square roots of the sum of an integer and *karaṇīs*

This is one of the finest contributions of Bhāskara, where he not only elaborates the algorithm for finding the square root of the sum of an integer and *karaṇīs*, but also states the existence conditions and the confirmative tests for the extracted square roots.

The algorithm

Bhāskara says [BīGa1980, p. 15]:

> वर्गे करण्या यदि वा करण्योः तुल्यानि रूपाण्यथवा बहूनाम् ।
> विशोधयेत् रूपकृतेः पदेन शेषस्य रूपाणि युतोनितानि ॥
>
> पृथक् तदर्धे करणीद्वयं स्यात् मूलेऽथ बह्वी करणी तयोर्या ।
> रूपाणि तान्येवमतोऽपि भूयः शेषाः करण्यो यदि सन्ति वर्गे ॥
>
> *vargaṃ karaṇyā yadi vā karaṇyoḥ stulyāni rūpāṇyathavā bahūnām |*
> *viśodhayet rūpakṛteḥ padena śeṣasya rūpāṇi yutonitāni ||*
>
> *pṛthak tadardhe karaṇīdvayaṃ syāt mūle'tha bahvī karaṇī tayoryā |*
> *rūpāṇi tānyevamato'pi bhūyaḥ śeṣāḥ karaṇyo yadi santi varge ||*
>
> Subtract one or more equivalent number of *karaṇīs* from the square of an integer [of the surd expression]. The square root of this difference is separately added to and subtracted from the [same] integer. The square root of the half of each term will be the two [required] *karaṇīs*. If any *karaṇīs* remain in the square [expression], then, assuming the greater *karaṇī* [out of the two obtained previously] as the new integer, the procedure is repeated [considering the remaining *karaṇīs*].

The condition for the selection of *karaṇīs*

Bhāskara says [BīGa1980, p. 16]:

> वर्गे करणीत्रितये करणीद्वितयस्य तुल्यरूपाणि ।
> करणीषट्के तिसृणां दशसु चतसृणां तिथिषु च पञ्चानाम् ।
> रूपकृतेः प्रोज्झ्य पदं ग्राह्यं चेदन्यथा न सत् क्वापि ।
>
> *varge karaṇītritaye karaṇīdvitayasya tulyarūpāṇi |*
>
> *karaṇīṣaṭke tisṝṇāṃ daśasu catasṝṇāṃ tithiṣu ca pañcānām |*
> *rūpakṛteḥ projjhya padaṃ grāhyaṃ cedanyathā na sat kvāpi |*
>
> That is, from three *karaṇīs* in the square [expression], select two; from six, three; from ten, four and from fifteen, five *karaṇīs* [for the subtraction]. If the difference from the square [of an integer thus obtained] is a perfect square number, then it should be considered, if not it will not be a proper [selection].

The confirmative tests

Bhāskara says [BīGa1980, p. 16]:

> उत्पत्स्यमानयैवं मूलकरण्याऽल्पया चतुर्गुणया ।
> यासामपवर्तः स्याद् रूपकृतेस्ता विशोध्याः स्युः ॥
>
> अपवर्ते या लब्धा मूलकरण्यो भवन्ति ताश्चापि ।
> शेषविधिना न यदि ता भवन्ति मूलं तदा तदसत् ॥
>
> *utpatsyamānayaivaṃ mūlakaraṇyā'lpayā caturguṇayā |*
> *yāsāmapavartaḥ syād rūpakṛtestā viśodhyāḥ syuḥ ||*
>
> *apavarte yā labdhā mūlakaraṇyo bhavanti tāścāpi |*
> *śeṣavidhinā na yadi tā bhavanti mūlaṃ tadā tadasat ||*
>
> Each of the *karaṇīs* [selected for the difference] must be divisible by four times the lesser root without remainder and each of the quotients must appear in the [final] answer. If not, then, the [derived] roots are invalid.

Bhāskara further suggests that [BīGa1980, p. 15]:

> ऋणात्मिका चेत् करणी कृतौ स्यात् धनात्मिकां तां परिकल्प्य साध्ये ।
> मूले करण्यावनयोरभीष्टा क्षयात्मिकैका सुधियाऽवगम्या ॥
>
> *ṛṇātmikā cet karaṇī kṛtau syāt dhanātmikāṃ tāṃ parikalpya sādhye |*
> *mūle karaṇyāvanayorabhīṣṭā kṣayātmikaikā sudhiyā'vagamyā ||*
>
> That is, if there is a negative term in the square [expression], then, assuming it to be positive, two roots should be obtained [following the said procedure]. One of the two should be optionally understood as negative by the one intelligent in computing the square root.

Algorithm:

- Let $E = R \pm \sqrt{k_1} \pm \sqrt{k_2} \pm \sqrt{k_3} \pm \ldots \pm \sqrt{k_m}$ such that R and each k_i is a positive integer.

  Condition 1: If $m = \dfrac{p(p+1)}{2}$ for $p = 1, 2, 3, 4, \ldots$ that is, $m = 1, 3, 6, 10, 15$ etc., then square roots of E exist.

- Bhāskara directs us to select 'x' *karaṇīs* out of m, such that for $m = 1, 3, 6, 10$, or 15, $x = 1, 2, 3, 4$ or 5 respectively and then find the difference. That is
$$R^2 - (k_1 + k_2 + k_3 + \cdots + k_x).$$

- Qualitatively, these 'x' *karaṇīs* are to be selected in such a way that the said difference is a perfect square number, y^2 (say).

- **Condition 2:** If such a selection is not possible for any of the ${}^mC_x$ combinations, then the square roots do not exist.

Note: Nārāyaṇa Paṇḍita in his *Bījagaṇitāvataṁsa* [BīGa1970, v. 50, p. 22], states the rule for this selection as:

> करणीखण्डमितिर्या द्विगुणा रूपाङ्घ्रियुक्तो मूलम् ।
> रूपदलेन विहीनं सङ्कलितपदं भवत्येव ।।
>
> *karaṇīkhaṇḍamitiryā dviguṇā rūpāṅghriyukto mūlaṃ* |
> *rūpadalena vihīnaṃ saṅkalitapadaṃ bhvatyeva* ||
>
> The number of karaṇīs [present in the given E] is multiplied by two [*dviguṇa*] and increased by one fourth of the unity [*rūpā-aṅghri*]. The square root [of the sum] is decreased by half of the unity [*rūpa-dal*]. [This number] is certainly, the number of terms in the sum.

The expression given in the verse above may be expressed as

$$x = \sqrt{2m + \frac{1}{4}} - \frac{1}{2}.$$

This author suggests a simpler form as: $x = \dfrac{+\sqrt{8m+1} - 1}{2}$. It may be noted that $(8m+1)$ will be a perfect square number only for $m = 1, 3, 6, 10, 15$ etc.

- Find $z_1 = \dfrac{R+y}{2}$ and $z_2 = \dfrac{R-y}{2}$. Then $\pm\sqrt{z_1}$ and $\pm\sqrt{z_2}$ will be the possible terms in the square root of E. Note that $z_1 > z_2$.
- If $m - x > 0$, then assuming $E = z_1\pm$ remaining $(m - x)$ *karaṇīs*, the said procedure is repeated and the newer values z_3, z_4 etc., are evaluated. This procedure is continued until $m - x = 0$.
- The square roots of E are any two values out of $\sqrt{z_1} \pm \sqrt{z_2} \pm \sqrt{z_3} \pm \sqrt{z_4} \ldots$.

Confirmative tests:

1. For $z_2 < z_1$, each of the 'x' *karaṇīs* selected for the subtraction must be divisible by $(4 \times z_2)$ without a remainder.
2. Each of the quotients must appear in the final answer for the square roots of E.

If any one of the above criteria fails, then the derived answers are incorrect.

- By considering the positive or negative sign for the surds involved in E, we judge the appropriate signs for the derived roots.

Now, we study seven examples, three of which are stated by Bhāskara, two by Nārāyaṇa Paṇḍita and two illustrations formulated by the author of this paper.

Illustration 4: Find the square roots of $E = 5 + \sqrt{24} + \sqrt{32}$ if they exist.

Here, $m = 2$, and so the first condition is not satisfied. Square roots do not exist.

Example 12: Bhāskara states that [BīGa1980, p. 17]:

चतुर्गुणाः सूर्यतिथीषु रुद्रनागर्तवो यत्र कृतौ करण्यः ।
स विश्वरूपा वद तत्पदं ते यद्यस्ति बीजे पटुताभिमानः ॥

caturguṇāḥ sūryatithīṣu rudranāgartavo yatra kṛtau karaṇyaḥ |
sa viśvarūpā vada tatpadaṃ te yadyasti bīje paṭutābhimānaḥ ||

If you are an expert algebraist, then tell the square roots of the expression with 13 as an integer and *karaṇīs* $12, 15, 5, 11, 8$ and 6 each multiplied by 4.

For, $E = 13 + \sqrt{48} + \sqrt{60} + \sqrt{20} + \sqrt{44} + \sqrt{32} + \sqrt{24}, m = 6$. Hence, the first condition is satisfied. Consider $13^2 - (48 + 60 + 20) = 41$, which is not a perfect square. The same is the case with the remaining $({}^6C_3 - 1 = 19)$ combinations. Thus, condition 2 is not satisfied and the square roots for E do not exist.

Example 13: Bhāskara states another similar example [BīGa1980, p. 17]:

वर्गं यत्र करण्यो दन्तैः सिद्धैर्गजैर्मिता विद्वन् ।
रूपैर्दशभिरुपेताः किं मूलं ब्रूहि तस्य स्यात् ॥

vargaṃ yatra karaṇyo dantaiḥ siddhairgajairmitā vidvan |
rūpairdaśabhirupetāḥ kiṃ mūlaṃ brūhi tasya syāt ||

Tell the square root of the square [expression] with 10 as an integer and *karaṇīs* 32, 24 and 8.

For $E = 10 + \sqrt{32} + \sqrt{24} + \sqrt{8}$, none of the differences like $10^2 - (32 + 24)$ are a perfect square number and hence square roots of E do not exist.

Example 14: Bhāskara states another example [BīGa1980, p. 17]:

वर्गे यत्र करण्यः तिथिविश्वहुताशनैश्चतुर्गुणितैः ।
तुल्या दशरूपाढ्याः किं मूलं ब्रूहि तस्य स्यात् ॥

varge yatra karaṇyaḥ tithiviśvahutāśanaiścaturguṇitaiḥ |
tulyā daśarūpāḍhyāḥ kiṃ mūlaṃ brūhi tasya syāt ||

Tell the square root, where the square [expression] has integer 10 added with the *karaṇīs* 15, 13 and 3 [each] multiplied by 4.

For $E = 10 + \sqrt{60} + \sqrt{52} + \sqrt{12}, \quad m = 3$. Hence, the first condition is satisfied. Further, $y^2 = 100 - (52 + 12) = 36$, is a perfect square number. Thus, the second condition is satisfied with $y = 6$. Hence, $z_1 = 8$ and $z_2 = 2$. But $(4 \times z_2)$ does not divide 52 and 12 without remainder. Thus, the first confirmative test fails, and therefore, square roots of E do not exist.

It may be noted here that Nārāyaṇa Paṇḍita also gives a similar example [BīGa1970, v. 25, p. 28]:

तिथिमनुरविविश्वककुभ्-नगसङ्ख्या कृतहताः करण्यश्चेत् ।
षोडशरूपसमेता यत्र कृतौ तत्र किं पदं कथय ।।
tithimanuraviviśvakakubh-nagasaṁkhyā kṛtahatāḥ karaṇyaścet |
ṣoḍaśarūpasametā yatra kṛtau tatra kiṃ padaṃ kathaya ||

Tell the square root of the square expression where integer is 16 along with surd numbers 15 (tithi), 14 (manu), 12 (ravi), 13 (viśva), 10 (kakubh) and 7 (naga) (each) multiplied by 4.

This amounts to the problem of finding the square root of

$$E = 16 + \sqrt{60} + \sqrt{56} + \sqrt{48} + \sqrt{52} + \sqrt{40} + \sqrt{28}.$$

Here, only two differences, $16^2 - (40+56+60)$ and $16^2 - (48+52+56)$ out of possible ${}^6C_3 = 20$, are perfect squares for which $z_1 = 13, z_2 = 3$. However, the surd numbers 40 and 56 are not divisible by $4z_1$ or $4z_2$. Hence, square roots of E do not exist.

Example 15: Bhāskara states another example [BīGa1980, p. 17]:

अष्टौ षट्पञ्चाशत् षष्टिः करणीत्रयं कृतौ यत्र ।
रूपैर्दशभिरूपेतं किं मूलं ब्रूहि तस्य स्यात् ॥
aṣṭau ṣaṭpañcāśat ṣaṣṭiḥ karaṇītrayaṃ kṛtau yatra |
rūpairdaśabhirūpetaṃ kiṃ mūlaṃ brūhi tasya syāt ||

Tell the square root, wherein the square has three *karaṇīs* 8, 56 and 60 along with an integer 10.

For $E = 10 + \sqrt{8} + \sqrt{56} + \sqrt{60}$, $y^2 = 100 - (8 + 56) = 36, z_1 = 8, z_2 = 2$ and $(4 \times z_2)$ divides 8 and 56 with quotients 1 and 7 respectively.

Now, we set $R = 8$ and then,

$$y^2 = 64 - 60 = 4, \quad y = 2, \quad z_3 = 5, \quad z_4 = 3$$

And $(4 \times z_4)$ divides 60 with quotient 5. Hence possible roots are $\pm\sqrt{2} \pm \sqrt{3} \pm \sqrt{5}$. However, quotients 1 and 7 do not appear in the possible answers. Thus, the second part of the confirmative test fails. Hence square roots of E do not exist.

Illustration 5: This author suggests the following example:

Find the square roots of

$$E = 34 + \sqrt{264} - \sqrt{168} - \sqrt{240} - \sqrt{308} - \sqrt{440} + \sqrt{280}.$$

We write $R = 34, k_1 = 264, k_2 = 168, k_3 = 240, k_4 = 308, k_5 = 440$ and $k_6 = 280$ treating negative *karaṇīs* as positive. Here $m = 6$ and hence the first condition is satisfied. We find

$$y^2 = R^2 - (k_1 + k_2 + k_3) = 484,$$

a perfect square number and $y = 22$. Now $z_1 = 28, z_2 = 6$ and $(4 \times 6 =)24$ divides 264, 168 and 240 with the quotients 11, 7 and 10 respectively. Hence ± 6 may be one of the terms in the answer. Now, we set $R = 28$ and find

$$y^2 = R^2 - (k_4 + k_5) = 36, \text{ and } y = 6.$$

Now, $z_3 = 17, z_4 = 11$, and $(4 \times 11 =)44$ divides 308 and 440 with quotients 7 and 10 respectively. Hence $\pm\sqrt{11}$ may be the second term in the answer.

Finally, we set $R = 17$ and find

$$y^2 = R^2 - k_6 = 9, y = 3, z_5 = 10, z_6 = 7,$$

and $(4 \times 7 =)28$ as well as $(4 \times 10 =)40$ divide 280 with quotients 10 and 7 respectively. Hence $\pm\sqrt{7}$ and $\pm\sqrt{10}$ may also be terms in the answer. As each of the quotients appear in the probable answers, the square roots of E are two values out of $\pm\sqrt{6} \pm \sqrt{11} \pm \sqrt{7} \pm \sqrt{10}$.

Now, 4 (6) (11) = 264, which is positive in E hence, we select $+\sqrt{6}$ and $+\sqrt{11}$ as the two values in the answer. Similarly, $4(6)(7) = 168$, which is negative hence, we select$-\sqrt{7}$ as the third value and $4(6)(10) = 240$, which is also negative hence, we select $-\sqrt{10}$ as the forth value for the first root. For the second root, the signs are reversed. Hence, square roots of E are

$$\sqrt{6} + \sqrt{11} - \sqrt{7} - \sqrt{10} \quad \text{and} \quad -\sqrt{6} - \sqrt{11} + \sqrt{7} + \sqrt{10}.$$

[Refer Bhāskara's suggestion in section 7.4 and algorithm in section 6].

Before we close this section, we would like to give a couple of more examples given by Bhāskara in his *Bījagaṇita* as illustrative examples:

[16] : $E = 17 + \sqrt{40} + \sqrt{80} + \sqrt{200}$. [BīGa1980, p. 17]
[17] : $E = 13 - \sqrt{84} + \sqrt{56} - \sqrt{28} - \sqrt{24} + \sqrt{12} - \sqrt{8}$. [BīGa1970, v. 21, p. 26]

Additional examples found in his Vāsanā are here under.

[18] : $E = 16 + \sqrt{120} + \sqrt{72} + \sqrt{60} + \sqrt{48} + \sqrt{40} + \sqrt{24}$. [*BB vāsanā*]
[19] : $E = 10 + \sqrt{24} - \sqrt{40} - \sqrt{60}$. [*BB vāsanā*].

8 Concluding remarks

Brahmagupta and Śrīpati present slightly variant methods for the addition and subtraction of the *karaṇīs* as compared to Bhāskara's methods. Later Nārāyaṇa Paṇḍita gives five methods which incorporate all the earlier methods [BīGa1970, vv. 26–30, pp. 13–14]. Bhāskara followed Brahmagupta and Śrīpati while discussing the methods for multiplication, square, division and square roots [Dat1993]. As regards to finding the square roots involving *karaṇīs*, it may be noted that Nārāyaṇa Paṇḍita, besides describing the method given by Bhāskara [BīGa1970, vv. 46–49, p. 22] also presents alternative methods.

Finally it may be mentioned that the concept of *yogaja-karaṇī* is a unique contribution of Bhāskara. Bhāskara is the first scholar to discuss the preconditions, limitations and the confirmative tests for the correct extraction of the square roots of the surd expression, with the following critical remark [BīGa2008, p. 63]:

यैरस्य मूलानयनस्य नियमो न कृतस्तेषामिदं दूषणम् ।

yairasya mūlānayanasya niyamo na kṛtasteṣāmidaṃ dūṣaṇam |

It is [indeed an] oversight on the part of those who have not [clearly] formulated these rules for extracting square roots.

Part IV

The Siddhāntaśiromaṇi: Gaṇitādhyāya

यत्र त्रातुमिदं जगज्जलजिनीबन्धौ समभ्युद्गते
ध्वान्तध्वंसविधौ विधौतविनमन्निःशेषदोषोच्चये ।
वर्तन्ते क्रतवः शतक्रतुमुखा दीव्यन्ति देवा दिवि
द्राङ् नः सूक्तिमुचं व्यनक्तु स गिरं गीर्वाणवन्द्यो रविः ॥

yatra trātum idaṃ jagaj jalajinībandhau samabhyudgate
dhvāntadhvaṃsavidhau vidhautavinamanniḥśeṣadoṣoccaye |
vartante kratavaḥ śatakratumukhā dīvyanti devā divi
drāṅ naḥ sūktimucaṃ vyanaktu sa giraṃ gīrvāṇavandyo raviḥ ||

When the friend of the lotuses (i.e., the Sun) rises for protecting this world, dispelling the darkness, clearing off all impurities of those who worship him with a pure mind, then commence the daily sacrificial rites (*kratu*) [on this earth] and Indra (*śatakratu*) and the other gods in heaven are pleased. May that sun, revered by gods, quickly open up our voice so that it can utter felicitous words.

Grahagaṇitādhyāya of Bhāskarācārya's *Siddhāntaśiromaṇi*

M. S. Sriram*

1 Introduction

The *Siddhāntaśiromaṇi* composed in 1150 CE by Bhāskarācārya is one of the most comprehensive treatises on Indian astronomy. It has two parts, namely, *Grahagaṇita* and *Golādhyāya*. *Grahagaṇita* expounds on all the standard calculations and algorithms in astronomy of Bhāskara's times, whereas *Golādhyāya* has the definitions, more fundamental issues (like the nature of the earth, the placement of stars and planets around it and so on), and the principles and theoretical details of these calculations. While the source verses of these two parts of the *Siddhāntaśiromaṇi* present the basic results and procedures, the *Vāsanābhāṣya* or *Mitākṣara* by Bhāskara himself gives a detailed exposition, almost like classroom lectures, on the entire subject. This includes details of proofs and justifications along with diagrams etc., in what are called the *upapattis*, as well as discussion on how theoretical concepts are linked with observations, constructions and use of instruments and so on.

In this article, we confine our attention to the explanations in the *Grahagaṇita* part of the *Siddhāntaśiromaṇi*. We discuss Bhāskara's treatment of some representative topics in the chapters on *Madhyamādhikāra* (mean longitudes), *Spaṣṭādhikāra* (true longitudes), and *Tripraśnādhikāra* (the three questions, namely, direction, location and time, essential for discussing the diurnal problems), in some detail.

In the *Bhagaṇādhyāya* section of the chapter on the *Madhayamādhikāra*, or 'mean longitudes', Bhāskara discusses the determination of the number of rev-

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olutions of the planets in a *kalpa*, by making observations using a *golayantra*. In section 2, we explain his method for finding the revolution numbers of the Sun and the Moon.

Section 3 is devoted to the instantaneous rate of motion of the planets. In Indian astronomical texts, the true longitude of a planet is obtained by applying some corrections to its mean longitude, which increases uniformly with time. Because of this, the daily motion of a planet which is the difference between the true longitudes at two successive sunrises, varies from day to day. This was recognised by the Indian astronomers even before Bhāskarācārya. In the chapter on *Spaṣṭādhikāra* or 'true longitudes' in *Grahagaṇita*, Bhāskara observes that it is only approximate to assign a fixed value to the daily motion throughout the course of a day, and one has to consider a '*tātkālikī*' ('instantaneous') value which varies during the course of the day, for better accuracy. For the Sun and the Moon, there is only one correction term, namely the *mandasaṃskāra* or the 'equation of centre', in the expression for the true longitude. In this case, he gives the correct expression for the 'instantaneous' rate of motion, which involves the identification of the cosine function as the derivative of the sine function. This has been discussed in detail elsewhere [Sri2014, pp. 221–236].

For the actual planets, that is, Mercury, Venus, Mars, Jupiter and Saturn, there is one more correction, called the '*śīghrasaṃskāra*'. In this case, a direct computation of the instantaneous rate of motion, would involve finding the derivative of the inverse sine function of a ratio of two variables. Bhāskara describes an ingenious method for finding the instantaneous rate of motion of these planets, which would involve only calculating the derivative of the sine function. Using this he obtains the 'stationary points' of the planets correctly (within the framework of his work), and discusses their retrograde motion.

The geometrical insights of Bhāskara come into full play in the chapter on *Tripraśnādhikāra* or 'the three questions (of direction, location and time)', which we consider in section 4. We take up the specific problem of finding the zenith distance of the Sun, given its declination, azimuth, and the latitude of the place, and explain one of his methods for solving the problem. In section 5, we make a few concluding remarks.

2 Revolution numbers of planets through observations

In the second chapter of *Grahagaṇita*, namely the *Madhyamādhikāra*, or the mean longitudes, there is a section, *Bhagaṇādhyāya* on the revolution numbers of the planets. The first six verses in this give the number of revolutions of the

Sun, Moon, Mars, Jupiter and Saturn, the *śīghroccas* of Mercury and Venus (which are the mean heliocentric planets), their apsides and nodes in a *kalpa* which is 1000 *mahāyuga*s, each made up of 43,20,000 years [SiŚi2005, p. 10]. The number of civil days in a *kalpa* is given in verse 9 [SiŚi2005, p. 14]. From these numbers, one can find the sidereal periods of the planets, and their apsides and nodes. How does one ascertain these numbers? Bhāskara states that the scriptures are the authority for these numbers. However, over a long period of time, the numbers may get changed in transmission, due to the mistakes generated by the scribe, the teacher, and the pupil. How does one remedy the situation?

Only the *upapatti* or the method determines the number of revolutions. However the number of revolutions cannot be determined directly, as the sidereal periods of the apsides and nodes and even the planet Saturn itself are long. Bhāskara discusses the strategy for finding the revolution numbers in the *Vāsanā* for verses 1–6. We will discuss his method for the sidereal period of the Sun and the Moon in the following.

First we give the general remarks made by him before he embarks upon the specific methods for the various revolution numbers [SiŚi2005, p. 11].

ग्रहाणां पूर्वगत्या गच्छतां कल्प एतावन्तो भगणा भवन्ति। तथा मन्दोच्चानां चलोच्चानां च प्राग्गत्या एतावन्तः पर्यया भवन्ति। तथा पातानां पश्चिमगत्या एतावन्तो भवन्ति।
अत्रोपपत्तिः। सा तु तत्तद्भाषाकुशलेन तत्तत्क्षेत्रसंस्थानज्ञेन श्रुतगोलेनैव ज्ञातुं[1] शक्यते नान्येन। ग्रहमन्दशीघ्रोच्चपाताः स्वस्वमार्गेषु गच्छन्त एतावतः पर्ययान् कल्पे कुर्वन्तीत्यत्रागम एव प्रमाणम्। स चागमो महता कालेन लेखकाध्यापकाध्येतृदोषैर्बहुधा जातः तदा कतमस्य प्रामाण्यम्। अथ यद्येवमुच्यते गणितस्कन्ध उपपत्तिमानेवागमः प्रमाणम्। उपपत्या ये सिध्यन्ति भगणाः ते ग्राह्याः। तदपि न। यतोऽतिप्राज्ञेन पुरुषेणोपपत्तिर्ज्ञातुमेव शक्यते। न तया तेषां भगणानामियत्ता कर्तुं शक्यते। पुरुषायुषाल्पत्वात्। उपपत्तौ तु ग्रहः प्रत्यहं यन्त्रेण वेध्यः॥

grahāṇāṃ pūrvagatyā gacchatāṃ kalpa etāvanto bhagaṇā bhavanti | tathā mandoccānāṃ caloccānāṃ ca prāggatyā etāvantaḥ paryayā bhavanti | tathā pātānāṃ paścimagatyā etāvanto bhavanti |

atropapattiḥ | sā tu tattadbhāṣākuśalena tattatkṣetrasaṃsthānajñena śrutagolenaiva jñātuṃ śakyate nānyena | grahamandaśīghroccapātāḥ svasvamārgeṣu gacchanta etāvataḥ paryayān kalpe kurvantītyatrāgama eva pramāṇam | sa cāgamo mahatā kālena lekhakādhyāpakādhyetṛdoṣairbahudhā jātaḥ tadā katamasya prāmāṇyam | atha yadyevamucyate gaṇitaskandha upapattimānevāgamaḥ pramāṇam | upapatyā ye sidhyanti bhagaṇāḥ te grāhyāḥ | tadapi na | yato'tiprājñena puruṣeṇopapattirjñātumeva śakyate | na tayā teṣāṃ bhagaṇānāmiyattā kartuṃ śakyate | puruṣāyuṣālpatvāt | upapattau tu grahaḥ pratyahaṃ yantreṇa vedhyaḥ ||

The planets make these many number of revolutions towards east in a *kalpa*. Similarly, the respective number of revolutions are [also] made [by their] *mandoccas* and *caloccas* towards east. Similarly, these many number [of revolutions] are made by [their] nodes, in the westerly direction.

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[1] The reading in the printed text is श्रोतुः we have emended it to ज्ञातुम्.

Here is the rationale. This [rationale] can only be understood by a person having a sound knowledge of language, geometry, and well versed in the [geometry of] spheres, and not by anyone else. Only scriptures (*āgama*) are the authority for the specified number of revolutions in a *kalpa* made by the planets, their *mandocca*, *śīghrocca*, and *pāta* in their own paths/orbits. Over a long period of time, large errors are accumulated because of the mistakes [that are generated] by the scribe, the teacher, and the pupil. Then which is authoritative? If it is told that the authority of the scriptures is established by the mathematical demonstrations (*gaṇitaskandha-upapatti*) only and that the number of revolutions obtained by the rationale (*upapatti*) are to be accepted, it is not so either. A very intelligent man can only understand the *upapatti* well. [However,] it is not possible [even for him] to determine the quantity/number (*iyattā*) [of revolutions] by means of it (i.e., *upapatti*). Because of the short lives of the human beings. In the *upapatti*, the planet is to be observed daily by an instrument (*yantra*).

[Translation by Sita Sundar Ram]

2.1 Sidereal year and the revolution number of the Sun

First, Bhāskara gives the method for finding the exact length of the solar year. On levelled ground, a circle is drawn and the east, west, north and south directions are marked. At the centre of the circle, a sharp pin is fixed and the direction of sunrise is noted on the circle using the pin. Let the sunrise correspond to the point S on a day when the Sun is close to equinox, but is south of the equator as illustrated in Figure 1 After 365 days, let S_1 be the point corresponding to the sunrise. It will be slightly south of S. On the next day, the sunrise point would be S_2 which would be slightly north of S. Let the length of the year be $365 + x$ days, where x is a fraction. x is found using the law of proportions. Now the arc S_1S would correspond to x days, whereas the arc S_1S_2 corresponds to one day. Hence

$$x = \frac{\text{arc } S_1S}{\text{arc } S_1S_2}.$$

This is explained thus [SiŚi2005, pp. 11–12].

अथ समायां भूमावभीष्टकर्कटकेन त्रिज्यामिताङ्कैरङ्कितेन वृत्तं दिगङ्कितं भगणांशैश्चाङ्कितं कृत्वा तत्र प्राचीचिह्नाद्दक्षिणतो नातिदूरे प्रदेश उत्तरेऽयने वृत्तमध्यस्थितेन कीलेन रवेरुदयो वेध्यः। ततोऽनन्तरं वर्षमेकं रव्युदया गणनीयाः। ते च पञ्चषष्ट्यधिकशतत्रय ३६५ तुल्या भवन्ति। तत्रान्तिमोदयः पूर्वोदयस्थानादासन्नो दक्षिणत एव भवति। तयोरन्तरं विगणय्य ग्राह्यम्। ततोऽन्यस्मिन् दिने पुनरुदयो वेध्यः। स तु पूर्वचिह्नादुत्तरत एव भवति। तदप्युत्तरमन्तरं ग्राह्यम्॥

atha samāyāṃ bhūmāvabhīṣṭakarkaṭakena trijyāmitāṅkairaṅkitena vṛttaṃ digaṅkitaṃ bhagaṇāṃśaiścāṅkitaṃ kṛtvā tatra prācīcihnāddakṣiṇato nātidūre pradeśa uttare'yane vṛttamadhyasthitena kīlena raverudayo vedhyaḥ | *tato'nantaraṃ*

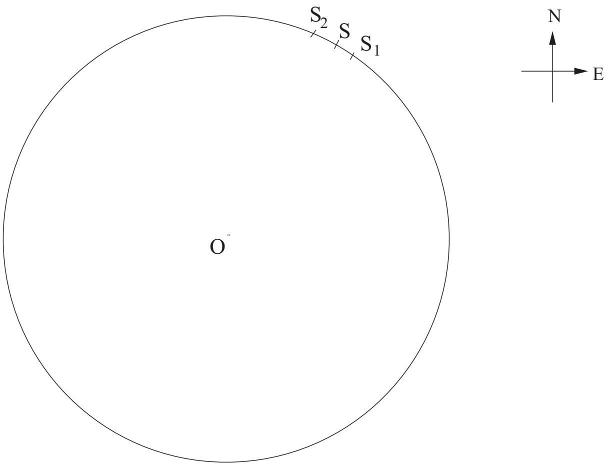

Figure 1: Observations for finding the length of the year.

varṣamekaṃ ravyudayā gaṇanī-yāḥ | te ca pañcaṣaṣṭyadhikaśatatraya 365 tulyā bhavanti | tatrāntimodayaḥ pūrvodayasthānādāsanno dakṣiṇata eva bhavati | tayorantaraṃ vigaṇayya grāhyam tato'nyasmin dine punarudayo vedhyaḥ | sa tu pūrvacihnāduttarata eva bhavati | tadapyuttaramantaraṃ grāhyam ||

On a flat ground, having drawn a circle by means of an optionally chosen compass marked with marks equal (in number) to the Radius (*trijyā*), and marking the four directions and degrees (fractions of a revolutions), observe (note down the position) the sunrise during the northward motion (*uttarāyaṇa*) when it is to the south of, but very close to the east point, by a sharp pin which is [kept] at the centre of the circle. Thereafter, the number of sunrises is to be counted for one whole year. They are equal to 365. There, the last sunrise is slightly away from the east point towards the south. Measure the difference between them (east point and the sunrise point). Thereafter observe the sunrise on the next (366^{th}) day. This will be towards north [slightly away] from the east point. Find that northern difference too.

ततोऽनुपातः। यद्यन्तरद्वितयकलाभिरेकीकृताभिः षष्टि ६० घटिका लभ्यन्ते तदा दक्षिणेनान्तरेण किमिति। अत्र लभ्यन्ते पञ्चदश घटिकास्त्रिंशत् पलानि सार्धानि द्वाविंशतिर्विपलानि १५। ३०। २२। ३०। आभिर्घटीभिः सहितानि पञ्चषष्ट्यधिकशतत्रयतुल्यानि सावनदिनान्येकस्मिन् रव्यब्दे भवन्ति ३६५। १५। ३०। २२। ३०। ततोऽनुपातः। यद्येकेन वर्षेणैतावन्ति कुदिनानि तदा कल्पवर्षैः किमिति। एवं ये लभ्यन्ते ते सावनदिवसा भवन्ति कल्पे। अथ तैरेव रवेर्वर्षान्तःपातिभिः कुदिनैश्चक्रकला लभ्यन्ते तदैकेन किमिति। फलं मध्यमा रविगतिरित्युपपन्नम्।

tato'nupātaḥ | yadyantaradvitayakalābhirekīkṛtābhiḥ ṣaṣṭi 60 ghaṭikā labhyante tadā dakṣiṇenāntareṇa kimiti | atra labhyante pañcadaśa ghaṭikāstriṃśat palāni sārdhāni dvāviṃśatirvipalāni 15 | 30 | 22 | 30 | ābhirghaṭībhiḥ sahitāni pañcaṣaṣṭyadhikaśatatrayatulyāni sāvanadinānyekasmin ravyabde bhavanti 365 | 15 | 30 | 22 | 30 | tato'nupātaḥ | yadyekena varṣeṇaitāvanti kudināni tadā kalpavarṣaiḥ kimiti | evaṃ ye labhyante te sāvanadivasā bhavanti kalpe atha taireva ravervarṣāntaḥpātibhiḥ

> *kudinaiścakrakalā labhyante tadaikena kimiti | phalaṃ madhyamā ravigatirityupapannam |*
>
> Now the proportion. If by the sum of the two differences (corresponding to the 365^{th} and 366^{th} days) is 60 *ghaṭikās* are obtained, then what is obtained for the southern difference? Here, [by applying rule of three] we obtain 15 *ghaṭikā*s, 30 *palas*, and $22\frac{1}{2}$ *vipalas*: 15 | 30 | 22 | 30. This added to 365 gives the number of civil days in a solar year as 365 | 15 | 30 | 22 | 30. Again, proportion. If these are the number of civil days in a [solar] year, then what would be the number of civil days in the years of a *kalpa*? The quantity obtained in this manner is the number of civil days in a *kalpa*. If by those civil days contained in a solar year, *kalās* of one revolution (i.e. 21600) are obtained, then what would be [the fraction of a revolution of the Sun] in one day? The result obtained is the rate of motion of the mean Sun. Thus is proved. [Translation by Sita Sundar Ram]

The numerical value of the solar year thus determined is given as 365 | 15 | 30 | 22 | 30 in sexagesimal units in the text. This is actually the numerical value of the sidereal year given in the *vāsanā* for verse 1 in the section on *Pratyabdaśuddhi* in *Madhyamādhikāra* of *Grahagaṇita* [SiŚi2005, p. 24]. However, the method that is given above in the text is clearly for the tropical year, which is less than 365 | 15 days. It is possible that the value presented is the one after adding the time unit corresponding to the motion of the equinoxes in one year. However, Bhāskara himself does not mention this. This point is discussed by Nṛsimha Daivajña in his *Vāsanāvārtika*, which is a super-commentary on the *Vāsanābhāṣya*. He mentions that the length of the year found from the above procedure is that of the *sāyanārka* (Sun with the *ayana*), that is, the tropical year, and that it becomes the sidereal year when the correction in time corresponding to the motion of the equinox is added [SiŚi1981, pp. 37–38].

2.2 Sidereal period of the Moon and its revolution number

Bhāskara prescribes the use of a *golayantra* (armillary sphere) to find the revolution numbers of the other planets and the associated points (like apsides and nodes), beginning with the Moon. It has a fixed celestial equator and an ecliptic which can be rotated around the polar axis. There is also a *vedhavalaya* to locate the planet, which is a movable ring and is a secondary to the ecliptic. There is a sight at the centre of the sphere.

The zero point of the Indian zodiac, which is the beginning point of the *Aśvinī nakṣatra* or the end point of the *Revatī nakṣatra* (which is also the end point of the *Mīna rāśi*) is located in the sky, and the point R corresponding to

that is marked on the ecliptic. Let the Moon be located at M at some instant on a particular day. Consider the point X, where the *vedhavalaya* on which the Moon is located, intersects the ecliptic. Then the arc RX is the (*nirayana*) longitude of the Moon, and the arc XM is the *vikṣepa* or the latitude of the Moon.

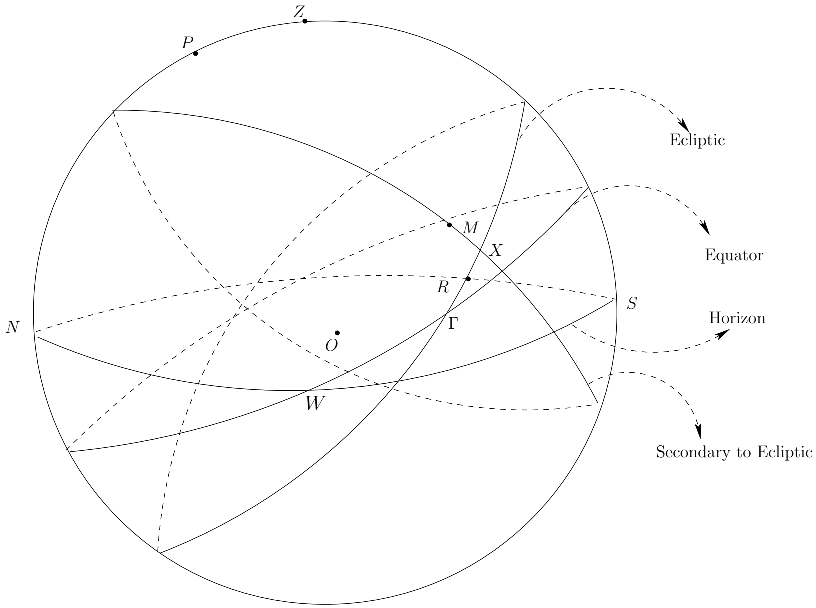

Figure 2: Finding the true longitude of the Moon using the *golayantra*.

Let X_1 and X_2 be the true longitudes of the Moon at the same *ghaṭī* or instant on two successive days. Then $X_2 - X_1$, which is the difference in the true longitudes is the true daily motion of the Moon. Now, the mean longitudes X_{10} and X_{20} on the two consecutive days can be found from the true longitudes, by an inverse process, as described in verse 45 in the chapter on "*Spaṣṭādhikāra*" or the "true longitudes".[2] Then $X_{20} - X_{10}$ is the daily motion of the mean Moon. From this, the sidereal period of the Moon can be computed. The number of revolutions of the Moon in a *kalpa* can then be determined from the rule of proportions. The method described in the *Vāsanābhāṣya* is as follows [SiŚi2005, p. 12]:

[2] [SiŚi2005, p. 59] The true longitude θ_{ms} can be found from the mean longitude θ_0, using (4) in the next section. The mean longitude, θ_0 can be obtained from the true longitude, θ_{ms} by an inverse process. Here, as the argument of the inverse sine function in (4) itself depends upon θ_0, one has to use an iterative procedure to find θ_0, where θ_{ms} is substituted for θ_0 in the argument, in the first approximation.

अथ चन्द्रभगणोपपत्तिः। तत्रादौ तावद् ग्रहवेधार्थं गोलबन्धोक्तविधिना विपुलं गोलयन्त्रं कार्यम्। तत्र खगोलस्यान्तर्भगोल आधारवृत्तद्वयस्योपरि विषुवद्वृत्तम्। तत्र च यथोक्तं क्रान्तिवृत्तं भगणांशाङ्कितं च बद्ध्वा कदम्बद्वयकीलकयोः प्रोतमन्यच्चलं वेधवलयम्। तच्च भगणांशाङ्कितं कार्यम्। ततस्तद्गोलयन्त्रं सम्यग्ध्रुवाभिमुखयष्टिकं जलसमक्षितिजवलयं च यथा भवति तथा स्थिरं कृत्वा गोलमध्यचिह्नगतया दृष्ट्या रेवतीतारां विलोक्य क्रान्तिवृत्ते यो मीनान्तस्थं रेवतीतारायां निवेश्य मध्यगतयैव दृष्ट्या चन्द्रं विलोक्य तद्वेधवलयं चन्द्रोपरि निवेश्यम्।

atha candrabhagaṇopapattiḥ | tatrādau tāvad grahavedhārtham golabandhoktavidhinā vipulam golayantram kāryam | tatra khagolasyāntarbhagola ādhāravṛttadvayasyopari viṣuvadvṛttam | tatra ca yathoktam krāntivṛttam bhagaṇāṃśāṅkitam ca baddhvā kadambadvayakīlakayoḥ protamanyaccalam vedhavalayam | tacca bhagaṇāṃśāṅkitam kāryam | tatastadgolayantram samyagdhruvābhimukhayaṣṭikam jalasamakṣitijavalayam ca yathā bhavati tathā sthiram kṛtvā golamadhyacihnagatayā dṛṣṭyā revatītārām vilokya krāntivṛtte yo mīnāntastham revatītārāyām niveśya madhyagatayaiva dṛṣṭyā candram vilokya tadvedhavalayam candropari niveśyam |

Now the rationale for the revolutions of Moon [is being stated]. First of all, in order to observe the planets, a huge armillary sphere (*golayantra*) has to be constructed/installed as per the rules specified in the chapter on *Golabandha* [in *Golādhyāya*]. There, [two] spheres, namely [outer] celestial (*khagola*) and inner astral (*bhagola*) [are to be constructed] with the support of two circles (*ādhāravṛtta-dvaya*) above which the celestial equator (*viṣuvadvṛtta*) [is also placed]. Similarly, as per the prescription, when one has constructed the ecliptic (*krāntivṛtta*) [which is one of the support circles] marked with degrees, the other ring of observation, which is movable and tied on two pins made of *Kadamba* wood, [is also constructed]. This is also marked with degrees. The *golayantra* is firmly fixed so that the axis-rod correctly points to the pole and the horizon-ring is adjusted with respect to the water level. Then, from a sight at the centre of the *gola*, the star *Revatī* is observed, and upon it, the end of *Mīna* on the ecliptic circle is fixed. Then looking at the Moon, the observation ring (*vedhavalaya*) is fixed over the Moon (such that it is a secondary to the ecliptic).

एवं कृते सति वेधवृत्तस्य क्रान्तिवृत्तस्य च यः संपातः तस्य मीनान्तस्य च यावदन्तरं तस्मिन् काले तावान् स्फुटचन्द्रो वेदितव्यः। क्रान्तिवृत्तस्य चन्द्रबिम्बमध्यस्य च वेधवृत्ते यावदन्तरं तावांस्तस्य विक्षेपः। ततो यावतीषु रात्रिगतघटिकासु वेधः कृतस्तावतीष्वेव पुनर्द्वितीयदिने कर्तव्यः। एवं द्वितीयदिने स्फुटचन्द्रं ज्ञात्वा तयोर्यदन्तरं सा तद्दिने स्फुटा गतिः।

evam kṛte sati vedhavṛttasya krāntivṛttasya ca yaḥ sampātaḥ tasya mīnāntasya ca yāvadantaram tasmin kāle tāvān sphuṭacandro veditavyaḥ | krāntivṛttasya candrabimbamadhyasya ca vedhavṛtte yāvadantaram tāvāṃstasya vikṣepaḥ | tato yāvatīṣu rātrigataghaṭikāsu vedhaḥ kṛtastāvatīṣveva punardvitīyadine kartavyaḥ | evam dvitīyadine sphuṭacandram jñātvā tayoryadantaram sā taddine sphuṭā gatiḥ |

Thus having been done, the distance [in degrees] from the intersection of this circle and the ecliptic to the end of *Mīna* is to be known as the true Moon (true longitude) at that time. The distance [in degrees] along the observation-circle (*vedhavṛtta*) measured from the ecliptic to the centre of the Moon, is the latitude (*vikṣepa*). Then, the experiment is repeated on the next day at the same time (*ghaṭikā*) of the night, as on the earlier night. Then, when one has determined the true Moon on the second day, their difference is the true rate of motion of the Moon on that day.

अथ तौ चन्द्रौ `स्फुटग्रहं मध्यखगं प्रकल्प्य' इत्यादिना मध्यमौ कृत्वा तयोरन्तरं सा मध्यमा चन्द्रगतिः। तयाऽनुपातः यद्येकेन दिनेन एतावती चन्द्रगतिस्तदा कुदिनैः किमित्येवं चन्द्रभगणा उत्पद्यन्ते। तथा चाह श्रीमान् ब्रह्मगुप्तः —

> ज्ञातं कृत्वा मध्यं भूयोऽन्यदिने तदन्तरं भुक्तिः।
> त्रैराशिकेन भुक्त्या कल्पग्रहमण्डलानयनम्॥

atha tau candrau sphuṭagrahaṃ madhyakhagaṃ prakalpya ityādinā madhyamau kṛtvā tayorantaraṃ sā madhyamā candragatiḥ | tayā'nupātaḥ yadyekena dinena etāvatī candragatistadā kudinaiḥ kimityevaṃ candrabhagaṇā utpadyante tathā cāha śrīmān brahmaguptaḥ —

> *jñātaṃ kṛtvā madhyaṃ bhūyo'nyadine tadantaraṃ bhuktiḥ |*
> *trairāśikena bhuktyā kalpagrahamaṇḍalānayanam ||*

Then the mean Moons are obtained (from the true Moons) from the procedure in which 'the true planet is imagined to be the mean object [in the first instance] etc.' (verse 45 in *Spaṣṭādhikāra* of *Grahagaṇita*), [SiŚi2005, p. 59] and by that the mean rate of motion of the Moon is to be found from the difference between them. Then, again the number of revolutions of the Moon is generated from the rule of proportions (*anupāta*): if the Moon travels this much in one day, then how much does it move in the number of civil days [in a *kalpa*]? That is what is said by Śrīmān Brahmagupta [BSS1966, p. 269]:

> After finding the mean (*madhya*) [on a day] and on the next day, the difference is the mean rate of motion (*bhukti*); the number of revolutions of the planet in a *kalpa* can be found from this rate using the rule of three (*trairāśika*).

[Translation by Sita Sundar Ram]

We will discuss how the apogee or the *mandocca* of the Moon is found through observations, towards the end of the next section.

3 The 'true' instantaneous rate of motion of planets

3.1 The instantaneous rate of motion of the *mandasphuṭa*

The term *mandasphuṭa* refers to the planet corrected for the equation centre. As the instantaneous rate of motion of a planet associated with the equation of centre or the *mandasaṃskāra* has been discussed elsewhere in detail [Sri2014], we will only summarise it in the following.

In ancient astronomy, epicycle and eccentric circle models were used to obtain the true longitude of a planet. In Figure 3, the 'mean' planet, P_0 moves at an uniform rate on the 'deferent' circle (solid circle in the figure) of radius R whose circumference is 21600, or $R = \frac{21600}{2\pi} \approx 3438$ around E, which is the *bhagolamadhya* (the centre of the astral sphere). $E'A$ is in the direction of the 'apside'. The '*mandasphuṭa*', P is situated on a small circle of radius, r around P_0 at P, such that P_0P is parallel to EA. Its longitude is also termed '*mandasphuṭa*' itself. $E\Gamma$ is in the direction of *meṣādi*, or the first point of Aries. This is the reference direction for measuring the longitudes.

We have an alternate picture of this in the eccentric circle model. Here, E' is a point at a distance of r from E, in the direction of EA. Then, the *mandasphuṭa*, P moves uniformly around E' at the same rate as P_0 around E in a circle of radius R (dashed circle in the figure), known as the '*pratimaṇḍala*' (eccentric circle), or the *grahavṛtta*. The motion of P around E would not be uniform. Here,

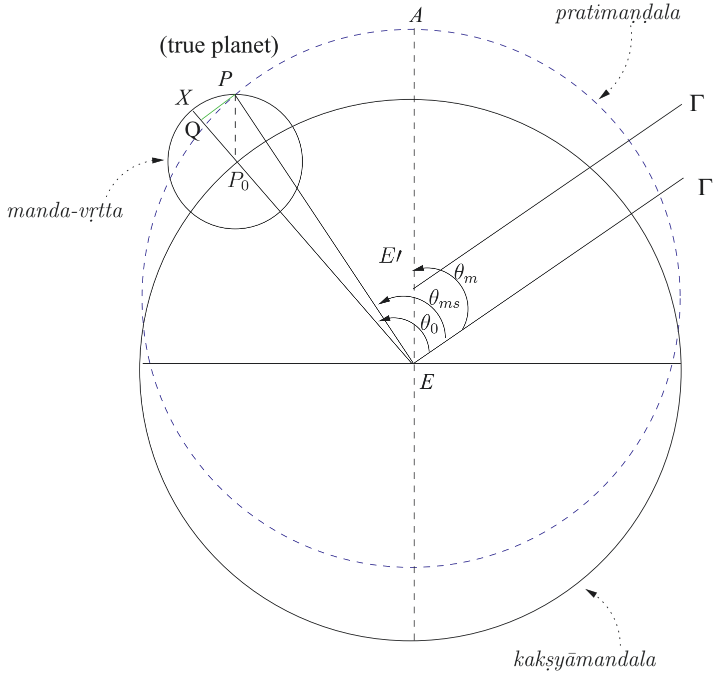

Figure 3: Epicycle and eccentric circle models for the equation of centre.

$$\begin{aligned}
\text{the mean longitude of the planet}, \theta_0 &= \mathit{\Gamma}\hat{E}P_0 = \mathit{\Gamma}\hat{E}'P, \\
\text{the longitude of the apside } (\textit{mandocca}), \theta_m &= \mathit{\Gamma}\hat{E}A, \\
\text{the mean anomaly } (\textit{mandakendra}), M &= A\hat{E}P_0 = \theta_0 - \theta_m, \\
\text{and the } \textit{mandasphuṭa}, \theta_{ms} &= \mathit{\Gamma}\hat{E}P.
\end{aligned}$$

The *mandakarṇa*, $K = EP$ is the distance between the planet (P) and the centre of deferent circle (E) and is given by:

$$K = \left[(R + r\cos M)^2 + (r\sin M)^2\right]^{\frac{1}{2}}. \tag{1}$$

The difference between the mean planet and the *mandasphuṭa* is $P\hat{E}P_0 = \theta_0 - \theta_{ms}$. Apart from the sign, this is the 'equation of centre'. It can be easily seen from the figure that

$$\sin(\theta_0 - \theta_{ms}) = \frac{r}{K}\sin M. \tag{2}$$

In most Indian texts, the epicycle radius associated with the equation of centre, r is proportional to the *mandakarṇa*, K and $\frac{r}{K} = \frac{r_0}{R}$, where r_0 is the specified constant value of the radius stated in the text [Shu1973, pp. 43–57]. Using this, we have

$$\sin(\theta_0 - \theta_{ms}) = \frac{r_0}{R}\sin M. \tag{3}$$

Thus the *mandasphuṭa*, θ_{ms} is given by

$$\theta_{ms} = \theta_0 - \sin^{-1}\left(\frac{r_0}{R}\sin M\right). \tag{4}$$

The *mandocca* (apside) is the apogee, and the *mandasphuṭa*, θ_{ms} is the 'true longitude' in the case of the Sun and the Moon (neglecting the second correction for the latter in some texts). For the *tārāgrahas* (actual planets), Mercury, Venus, Mars, Jupiter and Saturn, the *mandocca* is the aphelion, and the *mandasphuṭa* is essentially the heliocentric true longitude.

Now r_0 is much smaller than R. When θ_0, θ_{ms} and M are all in minutes, the arcsine should also be in minutes. Then, $\sin^{-1}\psi \approx R\psi$, when the argument of the arcsine, ψ is small, and we have,

$$\theta_{ms} = \theta_0 - \sin^{-1}\left(\frac{r_0}{R}\sin M\right) \approx \theta_0 - \frac{r_0}{R}\, R\sin M. \tag{5}$$

With our present knowledge, we can **now** say that

$$\frac{\mathit{\Delta}\theta_{ms}}{\mathit{\Delta}t} = \frac{\mathit{\Delta}\theta_0}{\mathit{\Delta}t} - \frac{r_0}{R}(\cos M)\frac{\mathit{\Delta}M}{\mathit{\Delta}t}, \tag{6}$$

where $\Delta\theta_0$, ΔM and $\Delta\theta_{ms}$ are the (small) changes in θ_0, M and θ_{ms} respectively in a small time-interval Δt. Also

$$\Delta(R\sin M) = (\cos M)\Delta M,$$

where ΔM is in minutes. Here, $\frac{\Delta\theta_{ms}}{\Delta t}$ is the rate of motion of the *mandasphuṭa*, and $\frac{\Delta\theta_0}{\Delta t}$ is the mean rate of motion. Choosing Δt to be one day, and denoting

(i) the daily motion of the mean longitude, $\frac{\Delta\theta_0}{\Delta t}$ by n_0,

(ii) the daily motion of the *mandasphuṭa*, $\frac{\Delta\theta_{ms}}{\Delta t}$ by n_{ms}, and

(iii) the daily motion of the anomaly, $\frac{\Delta M}{\Delta t}$ by n_M,

we have

$$n_{ms} = n_0 - \frac{r_0}{R}(\cos M)n_M. \tag{7}$$

This is essentially stated by Āryabhaṭa-II in the third chapter (v. 15) of his *Mahāsiddhānta* as [MaSi1910, p. 58]:

कोटिफलघ्नी भुक्तिर्गज्याभक्ता कलादिफलम् ॥

koṭiphalaghnī bhuktirgajyābhaktā kalādiphalam ||

The [mean] daily motion multiplied by the *koṭiphala* and divided by the radius (R) gives the correction in minutes etc., [to the mean rate of motion].

The *koṭiphala* is $r_0|\cos M|$. So, the verse gives the correction term in the rate of motion, namely, $\frac{r_0}{R}(\cos M)n_M$ correctly (the sign would depend on the quadrant in which M is situated).

Slightly different forms of the equation of centre, $\theta_{ms} - \theta_0$, and the correction to the rate of motion are given in *Laghumānasa* of Muñjālācārya, which appears to be the first text to consider the instantaneous rate of motion and use the cosine function as the derivative of the sine function, though they are not stated as such [LaMa1944, pp. 38–49].

In verses 36b, 37 and 38 of the chapter on *spaṣṭādhikāra* (true longitudes) in the *Grahagaṇitādhyāya* part of the *Siddhāntaśiromaṇi* [SiŚi2005, pp. 52–53], Bhāskara points out the need for finding an instantaneous (*tātkālikī*) rate of motion which varies from moment to moment, and gives the explicit expression for it, which is the same as in *Mahāsiddhānta*. This is explained in far greater detail in the *vāsanā* for the verses 36b–38 in *spaṣṭādhikāra* by Bhāskara himself [Sri2014, pp. 226–230].

3.2 Finding the apogee of the Moon, using the expression for the instantaneous rate of motion

Now, the mean longitude, θ_0, the *mandasphuṭa*, θ_{ms} and apogee, θ_M are all the same when $M = 0$. Also this happens when the rate of motion, given by

$$n = n_0 - \frac{r_0}{R} \cos M \; n_M,$$

has the least value, which happens when $\cos M$ has the maximum value at $M = 0$. This is noticed by Bhāskara, who uses this fact to describe how one can determine the apogee of the Moon, using the *golayantra*. This is discussed in the *vāsanā* for verses 1–6 in the *Bhagaṇādhyāya* section of *Madhyamādhikāra* [SiŚi2005, p. 12]:

> अथ चन्द्रोच्चस्य। एवं प्रत्यहं चन्द्रवेधं कृत्वा स्फुटगतयो विलोक्याः। यस्मिन् दिने गतेः परमाल्पत्वं दृष्टं तत्र दिने मध्यम एव स्फुटचन्द्रो भवति। तदेवोच्चस्थानम्। यत उच्चसमे ग्रहे फलाभावो गतेश्च परमाल्पत्वम्।
>
> *atha candroccasya | evaṃ pratyahaṃ candravedhaṃ kṛtvā sphuṭagatayo vilokyāḥ yasmin dine gateḥ paramālpatvaṃ dṛṣṭaṃ tatra dine madhyama eva sphuṭacandro bhavati | tadevoccasthānam yata uccasame grahe phalābhāvo gateśca paramālpatvam |*
>
> Now, the *Ucca* of the Moon: Thus the Moon's position and the true motion, are noted daily. The day on which the motion is the least, then the Mean Moon is also the True Moon. That is the position of the *mandocca*. Because at that position of *ucca*, there is no (*manda*) correction to the planet and the rate of motion is the least.

3.3 The true instantaneous rate of motion of the planets

We now consider the instantaneous rate of motion of the *tārāgraha*s, that is, Mars, Jupiter, Saturn (exterior planets), and Mercury and Venus (interior planets). For these planets, a second correction, namely, the *śīghra-saṃskāra* has to be applied, apart from the equation of centre, to obtain the true geocentric longitude. The *śīghra-saṃskāra* is equivalent to a conversion from the heliocentric to geocentric coordinates (reference). We describe an eccentric model for this correction as considered in many Indian texts, in the following [TaSa2011, pp. 500-503]. This is similar to the eccentric model for obtaining the *mandasphuṭa* from the mean longitude. Here the true longitude, *śīghra-sphuṭa* is obtained in a similar manner from the *mandasphuṭa*. Just as the

mandocca (apside) plays a major role in the application of *manda-saṃskāra*, so too the "*śīghrocca*" plays a key role in the application of the *śīghra-saṃskāra.*

In Figure 4, E is the centre of the Earth. S is the *śīghrocca* at a distance $ES = r_s$ from it. r_s is the *śīghra* epicycle radius. $\theta_s = \mathit{\Gamma}\hat{E}S$ is the longitude of *śīghrocca.* The "*kakṣyāmaṇḍala*" (deferent) is the circle of radius R with E as the centre. The "*pratimaṇḍala*" (eccentric circle) is the circle of radius R with S as the centre.

The *mandasphuṭa*, P_0 moves on the *kakṣyāmaṇḍala.* Its longitude is *mandasphuṭa*, $\theta_{ms} = \mathit{\Gamma}\hat{E}P_0$. Draw SP parallel to EP_0, with P on the *pratimaṇḍala.* Then the 'true' planet or *śīghrasphuṭa* is located at P, with *śīghrasphuṭa*, $\theta_t = \mathit{\Gamma}\hat{E}P$. θ_t is the true geocentric longitude of the planet. Note that $PP_0 = ES = r_s$.

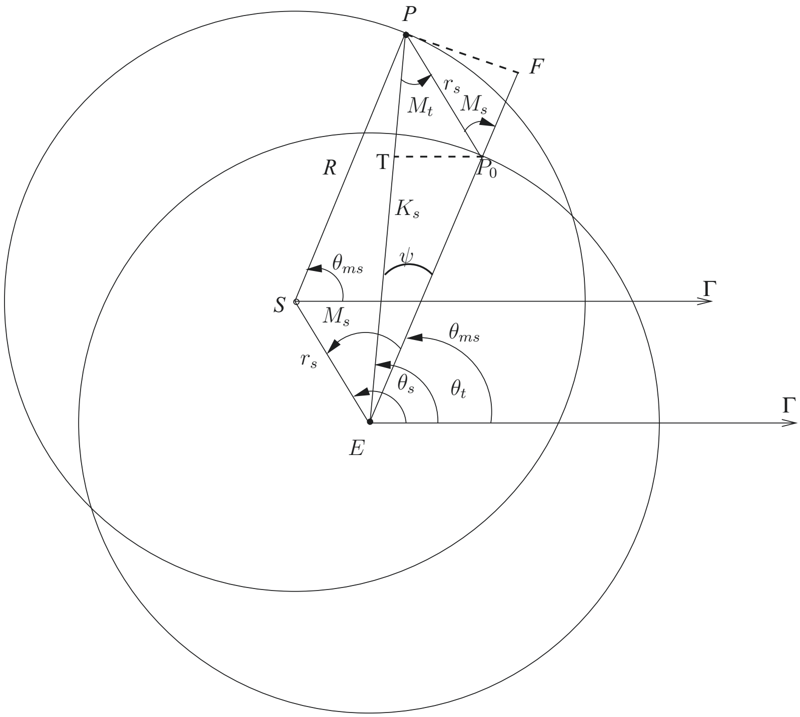

Figure 4: Finding the true planet (*śīghrasphuṭa*), P using a *śīghra* eccentric.

Now, it is seen that $\mathit{\Gamma}\hat{S}P = \mathit{\Gamma}\hat{E}P_0$ is also the *mandasphuṭa.* So, one can visualise the true planet as the *mandasphuṭa* moving on the *pratimaṇḍala*, with the *śīghrocca* as the centre.

The *śīghrakendra*, M_s is the difference between the *śīghrocca* and the *mandasphuṭa*, whereas the (*śīghra*) *sphuṭakendra*, M_t is the difference between the *śīghrocca* and the (*śīghra*) *sphuṭa*. EP, which is the distance between the centre of the Earth, E and the true planet, P, is the *śīghrakarṇa*, denoted by K_s. The *śīghraphala*, denoted by ψ is the difference between the *śīghrasphuṭa* (true planet) and the *mandasphuṭa*. Obviously, the *śīghrasphuṭa* is obtained by adding the *śīghraphala* to the *mandasphuṭa*.

It would be useful to summarise the definitions of the various quantities associated with the computation of the true planet:

$$\begin{aligned}
\textit{mandasphuṭa}, \theta_{ms} &= \Gamma\hat{E}P_0 = \Gamma\hat{S}P, \\
\textit{śīghrocca}, \theta_s &= \Gamma\hat{E}S, \\
\textit{śīghra}\text{ epicycle radius}, r_s &= ES = P_0P, \\
\textit{śīghrakendra}, M_s &= P_0\hat{E}S = \Gamma\hat{E}S - \Gamma\hat{E}P_0 = \theta_s - \theta_{ms}, \\
\textit{śīghrakarṇa}, K_s &= EP = [(R + r_s\cos M_s)^2 + r_s^2\sin^2 M_s]^{\frac{1}{2}}, \\
\textit{śīghrasphuṭa}, \theta_t &= \Gamma\hat{E}P, \\
\textit{śīghraphala}, \psi &= P_0\hat{E}P = \Gamma\hat{E}P - \Gamma\hat{E}P_0 = \theta_t - \theta_{ms}, \quad \text{and} \\
(\textit{śīghra})\ \textit{sphuṭakendra}, M_t &= P\hat{E}S = \Gamma\hat{E}S - \Gamma\hat{E}P = \theta_s - \theta_t.
\end{aligned}$$

From the figure, one can see that

$$PF = K_s\sin\psi = PP_0\sin M_s = r_s\sin M_s \tag{8}$$

$$\text{or,}\quad \sin\psi = \left[\frac{r_s}{K_s}\sin M_s\right] = \left[\frac{r_s}{K_s}\sin(\theta_s - \theta_{ms})\right]. \tag{9}$$

Therefore,

$$\theta_t = \theta_{ms} + \psi = \theta_{ms} + \sin^{-1}\left[\frac{r_s}{K_s}\sin(\theta_s - \theta_{ms})\right]. \tag{10}$$

Here it should be noted that, unlike in *mandasaṃskāra*, in *śīghrasaṃskāra*, the epicycle radius r_s is fixed. So, one would have to calculate K_s before computing the true planet. Also, $\frac{r_s}{R}$ is not small, so that the argument of the arcsine function $\chi = \frac{r_s}{K_s}\sin(\theta_s - \theta_{ms})$ is not small, and we cannot use the approximation, $\sin^{-1}\chi \approx \chi$.

The true velocity of the planet is $\frac{d\theta_t}{dt}$. If we calculate it from the above expression, it would be complicated. It involves the arcsine function whose argument depends upon K_s which also varies with time. Essentially, it would involve finding the derivative of the arcsine function, whose argument is a ratio of two functions.

Bhāskara gives an alternate expression for the 'true velocity' and gives an ingenious geometrical method of obtaining that expression which would

involve finding only the derivative of the sine function. Before discussing his method, we will first give a 'modern' derivation of the alternate expression for the 'true velocity'.

Now θ_t can also be written as

$$\theta_t = \Gamma \hat{E} P = \Gamma \hat{E} S - P \hat{E} S = \theta_s - M_t. \tag{11}$$

Therefore,

$$\frac{\Delta\theta_t}{\Delta t} = \frac{\Delta\theta_s}{\Delta t} - \frac{\Delta M_t}{\Delta t}. \tag{12}$$

It may be noted that,

$$\psi = \theta_t - \theta_{ms} = (\theta_s - \theta_{ms}) - (\theta_s - \theta_t) = M_s - M_t \tag{13}$$

$$\text{and} \quad P_0 T = r_s \sin M_t = R \sin\psi. \tag{14}$$

Taking the derivative of $r_s \sin M_t$, we obtain

$$r_s \cos M_t \frac{\Delta M_t}{\Delta t} = R\cos\psi \frac{\Delta\psi}{\Delta t} \tag{15}$$

Also,

$$\begin{aligned} r_s \cos M_t &= PT = EP - ET \\ &= K_s - R\cos\psi, \end{aligned} \tag{16}$$

$$\text{and} \qquad \frac{\Delta\psi}{\Delta t} = \frac{\Delta M_s}{\Delta t} - \frac{\Delta M_t}{\Delta t}. \tag{17}$$

Using the above in (15) we have,

$$(K_s - R\cos\psi)\frac{\Delta M_t}{\Delta t} = R\cos\psi\left(\frac{\Delta M_s}{\Delta t} - \frac{\Delta M_t}{\Delta t}\right). \tag{18}$$

Cancelling terms and dividing by K_s, we find that

$$\frac{\Delta M_t}{\Delta t} = \frac{R\cos\psi}{K_s}\left(\frac{\Delta M_s}{\Delta t}\right). \tag{19}$$

Now (12) reduces to

$$\frac{\Delta\theta_t}{\Delta t} = \frac{\Delta\theta_s}{\Delta t} - \frac{R\cos\psi}{K_s}\left(\frac{\Delta M_s}{\Delta t}\right). \tag{20}$$

Since the term, *gati* means the rate of motion, *sphuṭagati*, *śīghroccagati*, and (*śīghra*) *kendragati* are the rates of motion of the (*śīghra*) *sphuṭa*, θ_t ($\frac{\Delta\theta_t}{\Delta t}$),

śīghrocca, θ_s ($\frac{\Delta\theta_s}{\Delta t}$) and (*śīghra*) *kendra*, M_s ($\frac{\Delta M_s}{\Delta t}$), respectively. Also, $R\cos\psi$ is the *koṭi* of the *śīghraphala*, ψ. Hence,

$$sphuṭagati = śīghroccagati - \frac{R\cos(śīghraphala)}{karṇa} \times kendragati. \qquad (21)$$

Note that the *sphuṭagati* is negative when the magnitude of the second term on the RHS is greater than that of the first. Then the planet has a retrograde motion.

The expression for the *sphuṭagati* is what is stated by Bhāskara in verse 39 of *Spaṣṭādhikāra*, who also mentions when the retrograde motion occurs [SiŚi2005, p. 54]:

फलांशखाङ्कान्तरशिञ्जिनीघ्नी द्राक्केन्द्रभुक्तिः श्रुतिहृद्विशोध्या।
स्वशीघ्रभुक्तेः स्फुटखेटभुक्तिः शेषं च वक्रा विपरीतशुद्धौ॥ ३९ ॥

phalāṃśakhāṅkāntaraśiñjinīghnī
drākkendrabhuktiḥ śrutihṛdviśodhyā |
svaśīghrabhukteḥ sphuṭakheṭabhuktiḥ
śeṣaṃ ca vakrā viparītaśuddhau || 39 ||

Multiply the daily motion of the *drākkendra* (*śīghrakendra*) by the Rsine of the difference between *khāṅka* (90),[3] and the degrees of [*śīghra*] *phala* and divide by the hypotenuse. Subtract the result from the daily motion of its own *śīghrocca*. The remainder is the true motion of the planet. If the subtraction has to be done the other way (when the result is greater than the daily motion of its own *śīghrocca*), the planet is retrograde. [Translation by Sita Sundar Ram]

Note that the verse refers to Rsin(90° − *śīghraphala*), which is nothing but $R\cos$ (*śīghraphala*). Also it tells us that the planet's motion is retrograde, when the *śīghroccagati* is less than the "result" which is the magnitude of the second term.

We now give Bhāskara's brilliant geometrical derivation of the expression for the *sphuṭagati* as given in the *vāsanā* for the verse [SiŚi2005, pp. 54–55].

अत्रोपपत्तिः। अद्यतनश्वस्तनशीघ्रफलयोरन्तरं गतेः शीघ्रफलं भवति। तत्तु यथा मान्दं गतिफलं ग्रहफलवदानीतं तथा यद्यानीयते कृतेऽपि कर्णानुपाते सान्तरमेव स्यात् यथा धीवृद्धिदे। नहि केन्द्रगतिजमेव फलयोरन्तरं स्यात् किन्त्वन्यदपि। अद्यतनभुजफलश्वस्तनभुजफलान्तरे त्रिज्यागुणेऽद्यतनकर्णहृते यादृशं फलं न तादृशं श्वस्तनकर्णहृते। स्वल्पान्तरेऽपि कर्णे भाज्यस्य बहुत्वाद्बह्वन्तरं स्यादित्येतदानयनं हित्वा अन्यन्महामतिमद्भिः कल्पितम्। तद्यथा केन्द्रगतिरेव स्पष्टीकृता। तस्यां हि शीघ्रोच्चगतेः शोधितायां ग्रहस्य गतिः स्फुटैवावशिष्यत इति। अत्र स्फुटकेन्द्रगतिप्रदर्शनार्थं...

atropapattiḥ | adyatanaśvastanaśīghraphalayorantaraṃ gateḥ śīghraphalaṃ bhavati | tattu yathā māndaṃ gatiphalaṃ grahaphalavadānītaṃ tathā yadyānīyate

[3] This is because, in *Bhūtasaṅkhyā* system, *kha* is 0, and *aṇka* is 9.

kṛte'pi karṇānupāte sāntarameva syāt yathā dhīvṛddhide | nahi kendragatijameva phalayorantaraṃ syāt kintvanyadapi | adyatanabhujaphalaśvastanabhujaphalāntare trijyāguṇe'dyatanakarṇahṛte yādṛśaṃ phalaṃ na tādṛśaṃ śvastanakarṇahṛte | svalpāntare'pi karṇe bhājyasya bahutvādbahvantaraṃ syādityetadānayanaṃ hitvā anyanmahāmatimadbhiḥ kalpitam tadyathā kendragatireva spaṣṭīkṛtā | tasyāṃ hi śīghroccagateḥ śodhitāyāṃ grahasya gatiḥ sphuṭaivāvaśiṣyata iti | atra sphuṭakendragatipradarśanārthaṃ...

Here is the rationale [for the *sphuṭagati*]. The difference between the *śīghraphalas* of today's and tomorrow's would be the *gati* of the *śīghra-phala*. If the same procedure as in the case of the rate of motion of the equation of centre (*mandagatiphala*) is used for the *grahaphalas*, then there would be a difference in proportion in the hypotenuse (*karṇa*) as in the [text] *Dhīvṛddhida*. The difference in [*śīghraphala*s] are not only due to the variation in *kendragati*, but also due to some other [factor]. The result obtained by dividing the product of *trijyā* ('R') and the difference in the *bhujaphalas* of today's and tomorrow's by today's *karṇa*, would be different from the one when dividing by tomorrow's *karṇa*. Even if the variation of the hypotenuse (*karṇa*) is small, the error would be high, because of the largeness of the dividend (*bhājya*). Hence, this has been abandoned and a new method has been proposed by the scholars. It is such that- only the rate of motion of the *kendra* (*kendragati*) is corrected. This, when subtracted from the rate of motion of the *śīghrocca* would give the true rate of motion (*sphuṭagati*) of the planet. Here, in order to demonstrate the rate of motion of the true (*sphuṭa*) *kendra*...

[Translation by Venketeswara Pai, R.]

Bhāskara now explains how the *sphuṭakendragati* is obtained using the geometrical construction [SiŚi2005, pp. 55–56]:

अथ तन्मानज्ञानार्थमुपायः। यथा भूमध्याद्विनिःसृता कर्णरेखा कक्षावृत्ते अद्यतनमध्यमग्रहात् फलतुल्येऽन्तरे लग्ना। एवं प्रतिमण्डलमध्याद्विनिःसृता रेखा प्रतिवृत्तग्रहात् फलतुल्येऽन्तरे यथा लगति तथा कृता सती कर्णसमकलया तिष्ठति। तस्याः कर्णेन सह सर्वत्र तुल्यमेवान्तरं स्यादित्यर्थः। अथ तदवधित्वेन प्रतिमण्डले फलस्य ज्याऽङ्क्या यथा ज्याग्रं प्रतिवृत्तमध्यग्रहचिह्ने भवति। अथ केन्द्रगत्यधिकस्य च फलस्य ज्याऽङ्क्या। तयोर्जीवयोरन्तरं कर्णसूत्रात् तिर्यग्रूपं भवति। तदत्र गणितेन ज्याकरणवासनया सिध्यति।

atha tanmānajñānārthamupāyaḥ | yathā bhūmadhyādviniḥsṛtā karṇarekhā kakṣāvṛtte adyatanamadhyamagrahāt phalatulye'ntare lagnā | evaṃ pratimaṇḍalamadhyādviniḥsṛtā rekhā prativṛttagrahāt phalatulye'ntare yathā lagati tathā kṛtā satī karṇasamakalayā tiṣṭhati | tasyāḥ karṇena saha sarvatra tulyamevāntaraṃ syādityarthaḥ | atha tadavadhitvena pratimaṇḍale phalasya jyā'ṅkyā yathā jyāgraṃ prativṛttamadhyagrahacihne bhavati atha kendragatyadhikasya ca phalasya jyā'ṅkyā | tayorjīvayorantaraṃ karṇasūtrāt tiryagrūpaṃ bhavati | tadatra gaṇitena jyākaraṇavāsanayā sidhyati |

Now the strategy for finding that measure. In the manner in which the hypotenuse-line which issues from the centre of the earth touches the deferent circle at a distance equal to the *phala* from today's mean planet, in the same manner, when the line that issues from the centre of the eccentric circle, touches [the circle] where the point [of intersection] is at a distance equal to *phala* from the planet on the eccentric circle (*prativṛttagraha*), then the [line] lies, having the same (number of) of *kalās* as the hypotenuse (*karṇa*). That is, each point on the line is equidistant

from the *karṇa*. The chord (Rsine) of the *phala* should be marked on the eccentric circle upto it so that the tip of the chord might be upon the mark of the mean planet on the eccentric circle. So also, the Rsine of the *phala* increased by the *kendragati*. Their Rsine difference would be in the opposite direction from (i.e., perpendicular to) the hypotenuse. Here, that is obtained by calculation according to the rationale for obtaining Rsines.

शीघ्रफलस्य जीवायां क्रियमाणायां यद्भोग्यखण्डं तेन केन्द्रगतिर्गुण्या। शरद्विदस्रैर्भाज्या। लब्धं तु तयोर्जीवयोरन्तरं स्यात्। यतो ज्याग्रस्थेन भोग्यखण्डेन जीवाया उपचयः। अथ तस्य भोग्यखण्डस्य स्फुटीकरणम्। यदि त्रिज्यातुल्यया कोटिज्यया आद्यं भोग्यखण्डं तदा फलकोटिज्यया किमिति। एवं आद्यखण्डं फलकोटिज्या च केन्द्रगतेर्गुणौ शरद्विदस्रास्त्रिज्या च हरौ २२५। ३४३८॥

śīghraphalasya jīvāyāṃ kriyamāṇāyāṃ yadbhogyakhaṇḍaṃ tena kendragatirguṇyā | śaradvidasrairbhājyā | labdhaṃ tu tayorjīvayorantaraṃ syāt | yato jyāgrasthena bhogyakhaṇḍena jīvāyā upacayaḥ | atha tasya bhogyakhaṇḍasya sphuṭīkaraṇam | yadi trijyātulyayā koṭijyayā aadyaṃ bhogyakhaṇḍaṃ tadā phalakoṭijyayā kimiti | evaṃ ādyakhaṇḍaṃ phalakoṭijyā ca kendragaterguñau śaradvidasrāstrijyā ca harau 225 | 3438 ||

While obtaining the Rsines of the *śīghraphala*, the *bhogyakhaṇḍa* [which is obtained] has to be multiplied by the *kendragati*; and divided by *śaradvidasra* (225). [The result] obtained would be their Rsine difference, since an increase of Rsine would be value due to the *bhogyakhaṇḍa* at the end point of the chord. Therefore *bhogyakhaṇḍa* has to be corrected. If the first *bhogyakhaṇḍa* is obtained from the Rcosine which is numerically equal to R, then what would be [the *bhogyakhaṇḍa*] corresponding to the Rcosine of *phala*? Then, the *ādyakhaṇḍa* and the Rcosine of the *phala* are the multipliers, and 225 and 3438 are the divisors of the *kendragati*.

अथान्योऽनुपातः। यदि कर्णाग्रे एतावदन्तरं तदा त्रिज्याग्रे किमिति। लब्धं कक्षावृत्ते ज्यारूपं भवति। तस्य धनुःकरणे अल्पत्वाज्जीवा न शुध्यति किन्तु शरद्विदस्रा गुण आद्यखण्डं हरः स्यात्। तथा कृते दर्शनम्। गुणः। त्रि. फलको. आ. २२५॥ छेदाः। त्रि. क. आ. २२५। अत्र शरद्विदस्रतुल्ययोस्तथा त्रिज्यातुल्ययोस्तथाऽऽद्यखण्डतुल्ययोश्च गुणकभाजकयोः तुल्यत्वान्नाशे कृते केन्द्रगतेः फलकोटिज्यागुणः कर्णो हरः स्यात्। फलं तु स्फुटा केन्द्रगतिर्भवति। सा शीघ्रोच्चगतेः शोध्या। शेषं स्फुटा ग्रहगतिर्भवति।

athānyo'nupātaḥ | yadi karṇāgre etāvadantaraṃ tadā trijyāgre kimiti | labdhaṃ kakṣāvṛtte jyārūpaṃ bhavati | tasya dhanuḥkaraṇe alpatvājjīvā na śudhyati kintu śaradvidasrā guṇa ādyakhaṇḍaṃ haraḥ syāt | tathā kṛte darśanam guṇaḥ | tri. phalako. aā. 225 | chedā: | tri. ka. aā. 225 | atra śaradvidasratulyayostathā trijyātulyayostathā"dyakhaṇḍatulyayośca guṇakabhājakayoḥ tulyatvānnāśe kṛte kendragateḥ phalakoṭijyāguṇaḥ karṇo haraḥ syāt | phalaṃ tu sphuṭā kendragatirbhavati | sā śīghroccagateḥ śodhyā | śeṣaṃ sphuṭā grahagatirbhavati |

Now, the other proportionality. Now, if the difference at the end of *karṇa* is this much, then how much would it be at the end of *trijyā*? The result obtained would be the Rsine on the deferent circle(*kakṣyāvṛtta*). Since the value of the Rsine is too small, while finding the arc (sine-inverse) of that, there is no correction, but the multiplier is 225 and the divisor is *ādyakhaṇḍa*. Thus having been done, it can be seen that the multiplier is: $R \times R\cos \text{of}(\textit{phala}) \times \textit{ādyakhaṇḍa} \times 225$, and the divisor is: $R \times \textit{karṇa} \times \textit{ādyakhaṇḍa} \times 225$. Here, since the terms R, *ādyakhaṇḍa* and 225 appear both in the numerator and the denominator, having destroyed (cancelled)

them, the multiplier and the divisor of the *kendragati* are Rcosine of the (*phala*) and *karṇa* respectively. The result would be corrected (*sphuṭa*) *kendragati*. This has to be subtracted from the *śīghroccagati*, and the obtained result is *sphuṭa-grahagati*.

[Translation by Venketeswara Pai, R.]

Bhāskara's explanation of the expression for *sphuṭagati* uses the following geometrical construction.

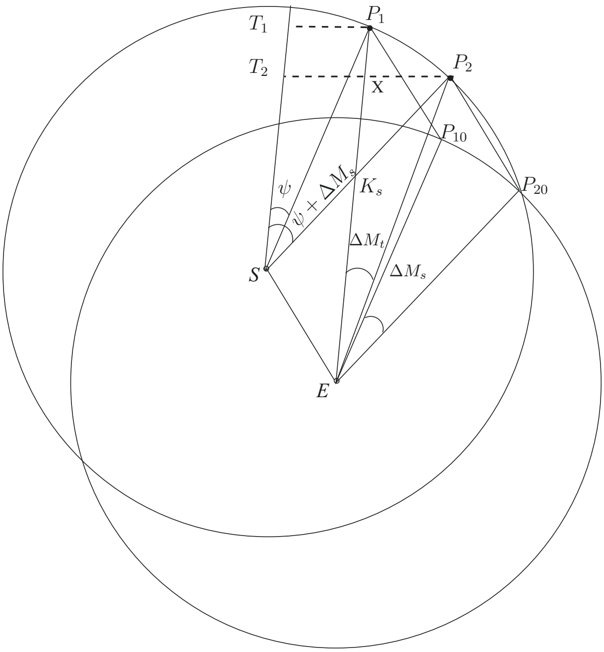

Figure 5: Bhāskara's geometrical construction for the true rate of motion (*sphuṭagati*) of a planet.

We consider the motion of the planet with respect to the *śīghrocca*, S. P_{10} and P_{20} are the *mandasphuṭas* on day 1 and 2, respectively. Similarly, P_1, P_2 are the *sphuṭas* (after *śīghra* correction) on day 1 and day 2. Draw a line from S parallel to EP_1. Let P_1T_1 and P_2T_2 be the perpendiculars from P_1 and P_2 on this line.

One can see that $P_1\hat{S}T_1 = \psi =$ *śīghraphala* on day 1. Now,

$$P_1\hat{S}P_2 = P_{10}\hat{E}P_{20} = \varDelta M_s = \text{Change in } (\textit{śīghrakendra}).$$

Then,

$$P_2\hat{S}T_2 = P_1\hat{S}T_1 + P_1\hat{S}P_2 = \psi + \varDelta M_s. \tag{22}$$

Also,

$$P_2X = P_2T_2 - P_1T_1 = R\sin(\psi + \varDelta M_s) - R\sin\psi. \tag{23}$$

Now,

$$R\sin(\psi + \varDelta M_s) - R\sin\psi = [R\sin(\psi + 225) - R\sin\psi] \times \frac{\varDelta M_s(\text{min.})}{225},$$

$$\text{and}\quad R\sin(\psi + 225) - R\sin\psi = \textit{ādyakhaṇḍa} \times \frac{R\cos\psi}{R}, \tag{24}$$

according to Bhāskara, where *ādyakhaṇḍa* $= R\sin(225) - R\sin(0)$ is the first Rsine-difference. Hence,

$$\begin{aligned} P_2X &= R\sin(\psi + \varDelta M_s) - R\sin\psi \\ &= \frac{\textit{ādyakhaṇḍa}}{225} \times \frac{R\cos\psi}{R} \times \varDelta M_s \text{ (minutes).} \end{aligned} \tag{25}$$

Now, $P_2\hat{E}P_1 = P_2\hat{E}S - P_1\hat{E}S = \varDelta M_t$: change in *sphuṭakendra*.

Bhāskara then observes that

$$\text{Arc } P_1P_2 = P_2X\ \frac{225}{\textit{ādyakhaṇḍa}} = \frac{R\cos\psi}{R}\ \varDelta M_s \text{ (minutes),} \tag{26}$$

is the change in the *sphuṭakendra* over a circle centred at E, whose radius is the *karṇa*, K_s. The change in the *sphuṭakendra* over a circle whose radius is the *trijyā*, R is found from the rule of proportions:

$$\varDelta M_t \text{ (minutes)} = \frac{R}{K_s} \times \frac{R\cos\psi}{R}\ \varDelta M_s \text{ (minutes).} \tag{27}$$

$$\text{Therefore,}\qquad \frac{\varDelta M_t}{\varDelta t} = \frac{R\cos\psi}{K_s}\frac{\varDelta M_s}{\varDelta t}, \tag{28}$$

which is the same as (19), from which follow (20) and (21) which is the desired result.

Till now, we have not mentioned what the physical significance of the *śīghrocca* and the *śīghra*-epicycle radius, r_s are. Actually, the *śīghrocca* is the mean Sun for the exterior planets (Mars, Jupiter and Saturn), and the mean heliocentric planet for the interior planets (Mercury and Venus) in all the Indian texts prior to *Tantrasaṅgraha* of Nīlakaṇṭha Somayājī [TaSa2011]. Also, $\frac{r_s}{R}$ is the ratio of the Sun-Earth and the Sun-planet distances for the exterior planets, and the inverse of that ratio for the interior planets. Before moving on to the next topic, it may be mentioned that all the considerations above in this subsection are applicable to both the exterior and interior planets.

3.4 Stationary points and retrograde motion of the planets

A planet has 'direct motion' when it moves eastwards in the background of stars. Then, its true rate of motion, $\frac{\Delta\theta_t}{\Delta t} > 0$. It has a *vakragati* or 'retrograde motion', when it moves westwards in the background of stars. Then, $\frac{\Delta\theta_t}{\Delta t} < 0$. The 'stationary points' correspond to the transition points in the planet's orbit, where the true rate of motion, $\frac{\Delta\theta_t}{\Delta t} = 0$.

In verse 41 of the *spaṣṭādhikāra*, Bhāskara discusses the retrograde motion and the values of the *śīghrakendra* corresponding to the stationary points of the five planets [SiŚi2005, p. 11].

द्राक्केन्द्रभागैस्त्रिनृपैः शरेन्द्रैः तत्त्वेन्दुभिः पञ्चनृपैस्त्रिरुद्रैः ।
स्याद्वक्रता भूमिसुतादिकानां अवक्रता तद्रहितैश्च भांशैः ॥ ४१ ॥

drākkendrabhāgaistrinṛpaiḥ śarendraiḥ stattvendubhiḥ pañcanṛpaistrirudraiḥ |
syādvakratā bhūmisutādikānāṃ avakratā tadrahitaiśca bhāṃśaiḥ || 41 ||

The planets, Mars, Mercury, Jupiter, Venus and Saturn will become retrograde when the anomaly has values *trinṛpa* 163 (*tri* =3, *nṛpa* =16), *śarendra* 145 (*śara* =5, *indra* =14), *tattvendu* 125 (*tattva=25*, *indu* =1), *pañcanṛpa* 165 (*pañca* =5, *nṛpa* = 16) and *trirudra* 113 (*tri* =3, *rudra* =11). Direct motion [will resume] when each [of this anomaly] is this [value] subtracted from 360°.

[Translation by Arkasomayaji [SiŚi1980]]

Bhāskara does not explain this verse in the *Vāsanābhāṣya*. We shall provide an explanation in the following. As the values stated in the above verse are constants and do not depend upon the *mandoccas* of the planets, it is obvious that the equation of centre is ignored and the *mandasphuṭa*, θ_{ms} is taken to be the mean planet, θ_0 itself. So, the *śīghrakendra*, M_s is taken to be $\theta_s - \theta_0$. Hence,

$$\frac{\Delta M_s}{\Delta_t} = \frac{\Delta\theta_s}{\Delta_t} - \frac{\Delta\theta_o}{\Delta_t}. \tag{29}$$

The stationary points correspond to

$$\frac{\Delta\theta_t}{\Delta t} = 0.$$

Using (20) and the above, we have

$$\frac{\Delta\theta_s}{\Delta t} - \frac{R\cos\psi}{K_s}\left(\frac{\Delta\theta_s}{\Delta t} - \frac{\Delta\theta_0}{\Delta t}\right) = 0. \tag{30}$$

It is obvious that the rate of motion of any object is proportional to the number of its revolutions, or *bhagaṇas* in a *kalpa* (or any unit of time, for that

matter). Hence,

$$\frac{\Delta\theta_s}{\Delta t} = c\, n_s, \quad \text{and} \quad \frac{\Delta\theta_0}{\Delta t} = c\, n_0,$$

where n_s and n_0 are the number of revolutions of the *śīghrocca* and the mean planet in a *kalpa*, respectively, and c is a constant. After cancelling out the common factor c, we can write (30) for the stationary points in terms of n_s and n_0 as,

$$n_s = \frac{R\cos\psi}{K_s}(n_s - n_0). \tag{31}$$

From the figure, it is clear that

$$K_s \cos\psi = R + r_s \cos M_s, \tag{32}$$

$$\text{and} \qquad K_s^2 = (R + r_s\cos M_s)^2 + (r_s \sin M_s)^2$$
$$= R^2 + r_s^2 + 2 r_s R \cos M_s. \tag{33}$$

Substituting these in the equation for the stationary point, and after some simplifications, we find that

$$\cos M_s|_{stationary} = -\frac{(r_s^2\, n_s + R^2\, n_0)}{(n_s + n_0) r_s R}$$
$$= -\frac{\left(\frac{r_s^2}{R^2} + \frac{n_0}{n_s}\right)}{\left(1 + \frac{n_0}{n_s}\right)\frac{r_s}{R}}. \tag{34}$$

If M_{st} is a solution of this equation in the second quadrant ($90° < M_{st} < 180°$), $360° - M_{st}$ in the third quadrant is also a solution. These two values correspond to the beginning and end of the retrograde motion. This is depicted in the following figure, where the angular variable is the *śīghrakendra*, M_s, and the horizontal line corresponds to $M_s = 0$. Then, we have direct motion from $P_0 \to P_{R1}$, retrograde motion from $P_{R1} \to P_{R2}$, and direct motion again from $P_{R2} \to P_0$. It is the value of M_{st} which is given in verse 41, for each planet.

For the exterior planets, the *śīghrocca* is the mean Sun, and its number of revolutions in a *kalpa*, n_s is 4320000000. The mean planet is the mean heliocentric planet and the number of revolutions, n_0 in a *kalpa* of Mars, Jupiter and Saturn are given to be 2296828522, 364226455, 146567298 respectively, in the text.

For the interior planets, the *śīghrocca* is the mean heliocentric planet, and the number of revolutions of this in a *kalpa*, n_s for Mercury and Venus are given to be 17936998984, 7022389492, in the text. The mean planet is the mean Sun for them, and hence $n_0 = 4320000000$.

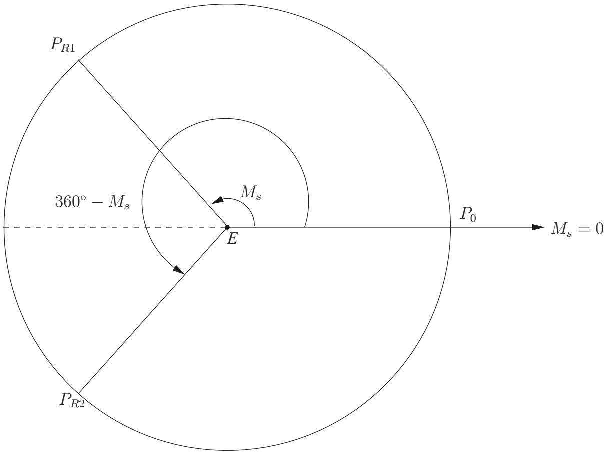

Figure 6: Direct ($P_0 \to P_{R1}$ and $P_{R2} \to P_0$), and retrograde ($P_{R1} \to P_{R2}$) motions of a planet: P_{R1} and P_{R2} are the stationary points.

The values of the ratios of the *śīghra* epicycle radii and the radius of the deferent, that is, $\frac{r_s}{R}$ for Mercury, Venus, Mars, Jupiter and Saturn are given to be $\frac{132}{360}, \frac{238}{360}, \frac{243\frac{2}{3}}{360}, \frac{68}{360}$, and $\frac{40}{360}$, respectively.

We have computed the values of M_{st} using these values of r_s, n_0 and n_s for each planet, and compared them with the values stated by Bhāskara (see Table 1). There is a remarkable agreement between the computed and stated values.

Table 1: The stationary points of the planets.

	Grahagaṇita value of n_0/n_s	*Grahagaṇita* value of r_s/R	Computed value of M_{st}	Stated value of M_{st}
Mercury	.2408	.3667	145.57°	145°
Venus	.6152	.7167	167.2°	165°
Mars	.5317	.6769	162.69°	163°
Jupiter	.0843	.1889	125.86°	125°
Saturn	.0339	.1111	113.75°	113°

4 Zenith distance of the Sun for an arbitrary azimuth

Let z and A be the zenith distance and the azimuth of the Sun (S) when its declination is δ at a place with latitude ϕ, as shown in Figure 7.

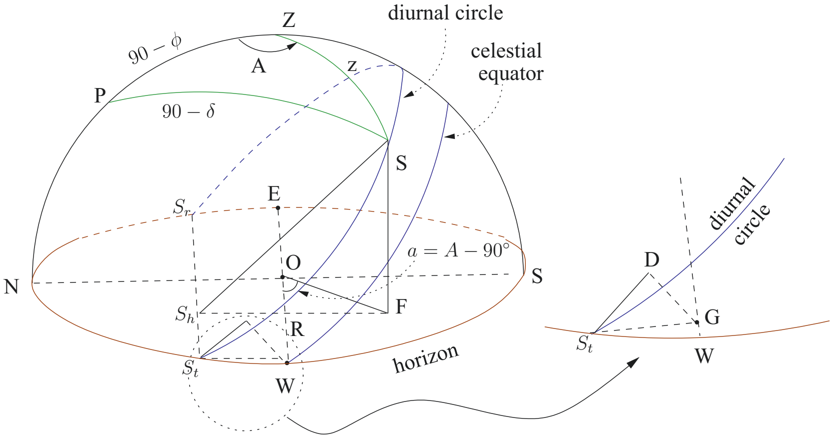

Figure 7: The zenith distance, z for a northern declination and $A > 90°$.

Here A is the angle between the north-south or the meridian circle, and the vertical passing through S ($P\hat{Z}S$). In the Indian texts, the azimuthal angle, termed the *digaṃśa*, is the angle between the vertical passing through S and the prime vertical, which we denote by a. Clearly, $A = 90° \pm a$, when $0° \leq A \leq 180°$. In the figure, $A = 90° + a$.

Using the modern cosine formula applied to the spherical triangle ZPS, where the sides are $PZ = 90 - \phi, ZS = z, PS = 90 - \delta$, we have

$$\cos(90-\delta) = \cos(90-\phi)\cos z + \sin(90-\phi)\sin z \cos A, \tag{35}$$

$$\text{or,} \quad \cos z \sin\phi = \sin\delta \pm \sin z \sin a \cos\phi \quad (\because A = 90 \pm a). \tag{36}$$

Now δ can be found directly from ϕ, z, and a from the above equation. However z cannot be found directly in terms of δ, ϕ, and a, as both $\cos z$ and $\sin z$ appear in the equation. One would have to solve a quadratic equation for $\sin z$, after squaring both sides and using $\cos^2 z = 1 - \sin^2 z$.

4.1 *Śaṅku, bhujā, agrā and śaṅkutala*

In Figure 7, the Sun rises at S_r, moves along the diurnal circle and sets at S_t. If we assume that Sun's declination δ is constant through the day, the 'rising-setting' line, S_rS_t would be parallel to the east-west line. From S_t, draw S_tG perpendicular to the east-west line meeting it at G. S_tG is the "*arkāgrā*" or just "*agrā*". It is the distance between the 'rising-setting' line and the east-west line.

Now the plane of the diurnal circle is inclined at an angle $90 - \phi$ with the horizon. From G draw GD perpendicular to the plane of the diurnal circle meeting it at D. Join S_tD, which would be perpendicular to GD. Clearly, $D\hat{S_t}G = 90 - \phi$ and $D\hat{G}S_t = \phi$. S_tDG is a latitudinal triangle (a right-angled triangle with the latitude as one of the angles). Now $GD = |R\sin\delta|$. Hence, $agr\bar{a} = S_tG = \left|R\frac{\sin\delta}{\cos\phi}\right|$.

From S, draw SF perpendicular to the plane of the horizon. In Indian astronomy texts, $SF = R\cos z$ is called "*śaṅku*" or gnomon and $OF = R\sin z$ is called "*dṛgjyā*." Draw RF perpendicular to the east-west line. $RF = R\sin z\sin a$ and is called the "*bhujā*". It is the distance between the base of the *śaṅku* and the east-west line.

Extend FR to meet the rising-setting line perpendicularly, at S_h. S_hF is the distance between the base of the *śaṅku*, F and the rising-setting line, and is called the "*śaṅkutala*". SS_hF is a latitudinal triangle, with $S\hat{S_h}F = 90° - \phi$. Hence, the *śaṅkutala*, $S_hF = SF\frac{\sin\phi}{\cos\phi} = R\cos z\frac{\sin\phi}{\cos\phi}$.

Multiplying (36) by R and dividing by $\cos\phi$, and rearranging terms, we find:

$$\pm R\sin z\sin a = R\cos z\frac{\sin\phi}{\cos\phi} - \frac{R\sin\delta}{\cos\phi}. \tag{37}$$

Now Figure 7 corresponds to the case of a northern declination, that is, $\delta = |\delta|$, and $A = 90° + a$, when we have to take the positive sign in the LHS of the above equation. Hence,

$$bhuj\bar{a} = \acute{s}a\dot{n}kutala - agr\bar{a}, \quad \delta \text{ north}, \quad \text{and } A = 90° + a. \tag{38}$$

Figure 8 depicts the situation when the declination is north, and $A = 90° - a$, in which case, we have to take the negative sign in the LHS of the equation, and

$$bhuj\bar{a} = agr\bar{a} - \acute{s}a\dot{n}kutala, \quad \delta \text{ north}, \quad \text{and } A = 90° - a. \tag{39}$$

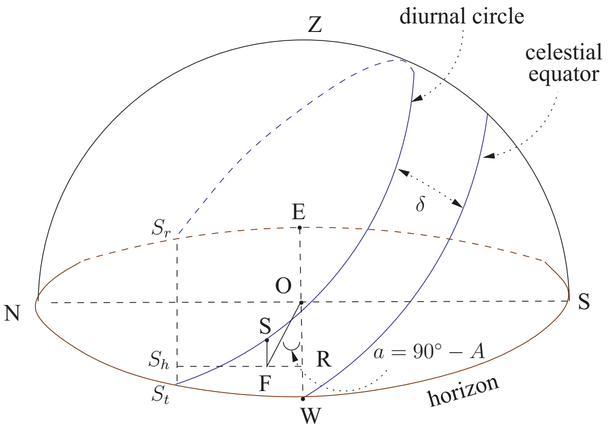

Figure 8: The zenith distance, z for a northern declination and $A < 90°$.

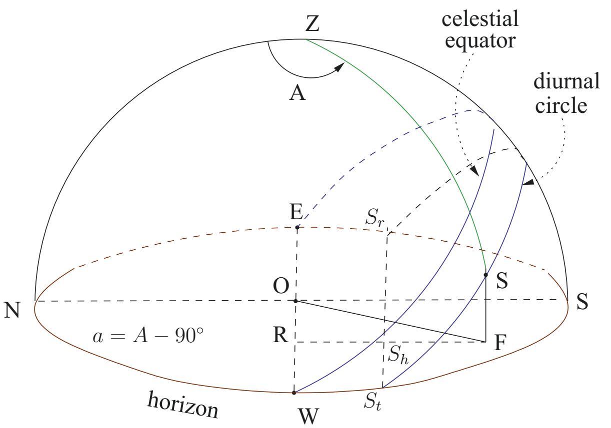

Figure 9: The zenith distance, z for a southern declination and $A > 90°$.

When the declination is south, $\delta = -|\delta|$, and *agrā* $= -\frac{R \sin \delta}{\cos \phi}$. Here $A = 90° + a$, necessarily. In this case, shown in Figure 9,

$$bhuj\bar{a} = \acute{s}a\dot{n}kutala + agr\bar{a}, \quad \delta \text{ south.} \tag{40}$$

Actually, these relations follow from the definitions of *bhujā*, *śaṅkutala*, and *agrā* and the geometry of the problem, as evident from the figures. They are equivalent to the cosine formula for the side PS.

In the *upapatti* for verse 30, Bhāskara states these relations [SiŚi2005, p. 57]:

स्वाग्रास्वशङ्कुतलयोर्याम्यगोले योगः सौम्ये त्वन्तरं भुजो भवति ।

svāgrāsvaśaṅkutalayoryāmyagole yogaḥ saumye tvantaraṃ bhujo bhavati |

The sum of the *agrā* and *śaṅkutala* in the southern hemisphere, and their difference in the northern hemisphere gives the *bhuja*.

In the *Tantrasaṅgraha* of Nīlakaṇṭha Somayājī also, these relations are stated in the chapter on *Chāyāprakaraṇaṃ*, though the nomenclature is slightly different [TaSa2011, pp. 185–189]. In this work, the words *mahābāhu* and *śaṅkvagrā* are used for *bhujā* and *śaṅkutala* respectively.

4.2 Finding the zenith distance

The equation relating the *bhujā*, *agrā* and *śaṅkutala* is only the first step in solving for the *śaṅku*, $R\cos z$, and the zenith distance, z from that. Actually, Bhāskara casts the equation in terms of a '*chāyā–karṇa*' or the 'shadow hypotenuse', and solves it.

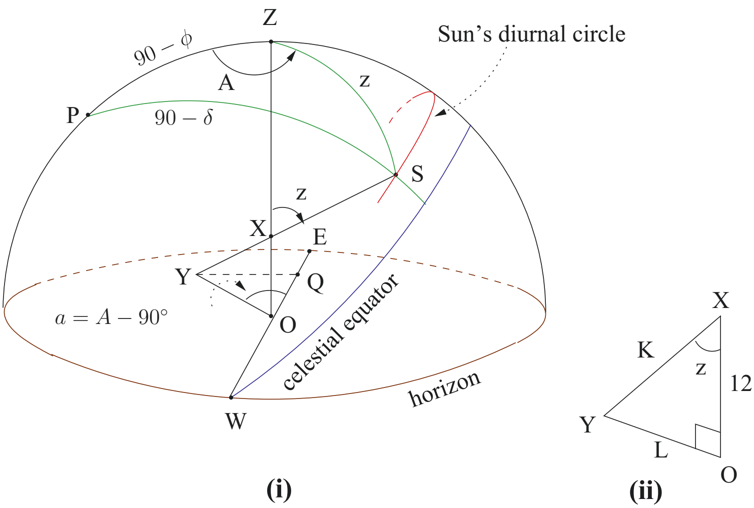

Figure 10: The 12-digit gnomon (*dvādaśāgulaśaṅku*), the shadow (*chāyā*), S and the shadow-hypotenuse (*chāyākarṇa*), K.

In Figure. 10, we consider the same situation as in Figure 7, when the Sun has a declination δ, zenith distance z, and azimuth A, for a location with latitude ϕ. OX is a 12-digit gnomon, (*dvādaśāṅgulaśaṅku*): $OX = 12$. Then

$$OY = S = 12\,\frac{\sin z}{\cos z},$$

is the shadow of this gnomon, or the "*chāyā*", and $K = \frac{12}{\cos z}$ is the "*chāyā-karṇa*", or the shadow-hypotenuse. Draw YQ perpendicular to the east-west line. It is given by

$$YQ = B = S\sin a = 12\frac{\sin z}{\cos z}\sin a,$$

where $A = 90° \pm a$, is called the "*chāyābhujā*". In the figure $A = 90° + a$. Note that

$$K^2 = S^2 + 12^2 = S^2 + 144.$$

Multiplying (37) by $K = \frac{12}{\cos z}$, and dividing by R, we find

$$\pm(\mathit{ch\bar{a}y\bar{a}bhuj\bar{a}},\ \ B) = 12\frac{\sin\phi}{\cos\phi} - \frac{K}{R}\frac{R\sin\delta}{\cos\phi}. \tag{41}$$

On the equinoctial day, when $\delta = 0$, the Sun is on the equator, Hence, the *chāyābhujā*, which is the distance between the tip of the shadow and the east-west line is $s = 12\frac{\sin\phi}{\cos\phi}$. This is called the "*palabhā*" and is a constant for a given latitude. Hence, on the equinoctial day, the tip of the shadow of the gnomon moves on a straight line parallel to the east-west line, at a distance equal to the *palabhā*. Note that the *chāyābhujā* is the shadow itself at noon, when the Sun is on the meridian, and $a = 90°$. Hence, the *palabhā*, $s = 12\frac{\sin\phi}{\cos\phi}$, is the equinoctial mid-day shadow.

Now, denoting the *agrā*, $|\frac{R\sin\delta}{\cos\phi}|$ by $\mathcal{A}$, and multiplying (41) by R, we find that

$$B\cdot R = s\cdot R \sim K\cdot\mathcal{A}, \ \ (\delta\ \text{ north}) \tag{42}$$

$$B\cdot R = s\cdot R + K\cdot\mathcal{A}. \ \ (\delta\ \text{ south}) \tag{43}$$

Now, $B\cdot R = S\sin a\cdot R = S\cdot D$, where $D = R\sin a$ is the *digjyā*. Squaring the equations above, and noting that $K^2 = S^2 + 144$, we obtain the following equation for K:

$$(K^2 - 144)D^2 = K^2\mathcal{A}^2 \pm 2K\mathcal{A}sR + s^2R^2. \tag{44}$$

In the above equation " + " is for δ south, and " − " for δ north. After rearranging the terms, and dividing by $D^2 - \mathcal{A}^2$ we have the following quadratic equation for K:

$$K^2 \mp 2K\frac{\mathcal{A}sR}{D^2 - \mathcal{A}^2} = \frac{s^2R^2 + 144D^2}{D^2 - \mathcal{A}^2}. \tag{45}$$

Defining two variables x and y through the relations:

$$x = \frac{s^2R^2 + 144D^2}{D^2 - \mathcal{A}^2}, \quad \text{and} \quad y = \frac{\mathcal{A}sR}{D^2 - \mathcal{A}^2}, \tag{46}$$

the formal solutions of the quadratic equation for K are given by:

$$\begin{aligned} & K = y \pm \sqrt{x + y^2}, \quad \text{for } \delta \text{ south,} \\ \text{and} \qquad & K = -y \pm \sqrt{x + y^2}, \quad \text{for } \delta \text{ north.} \end{aligned} \tag{47}$$

However, for the physical solutions, the zenith distance $z \leq 90°$, and K should be positive. We consider the various cases now.

δ is south: In this case, from Figure 9, it is clear that the *digjyā*, D is necessarily greater than the *agrā*, $\mathcal{A}$. Then x is positive, and $\sqrt{x + y^2} \geq y$. Hence, for positive K, only the "+" sign in front of the square root is permissible, and thus we can only consider the solution:

$$K = y + \sqrt{x + y^2} \ . \tag{48}$$

δ is north, and $D > \mathcal{A}$: In this case also, only the '+' sign in front of the square root is permissible, and

$$K = -y + \sqrt{x + y^2}. \tag{49}$$

δ is north, and $D < \mathcal{A}$: In this case, both x and y are negative, $-y = |y|$, and $\sqrt{x + y^2} < |y|$. Then both the solutions result in positive K and

$$K = -y \pm \sqrt{x + y^2}. \tag{50}$$

The two solutions correspond to the location of the Sun south and north of the prime vertical with the same value of a, and hence the same value of the *digjyā*, D, but different values of A, namely $90° \pm a$.

In the *upapatti* for verses 49, 50 and 51 of *Tripraśnādhikāra*, Bhāskara explains how the quadratic equation for the *chāyākarṇa* is arrived at, and also gives the solutions, as above [SiŚi2005, pp. 85–86].

अत्रोपपत्तिर्बीजगणितप्रक्रियया। तत्राव्यक्तं याकारोपलक्षितं त्रिज्याग्रादिका आद्याक्षरोपलक्षिताः कृत्वा बीजप्रक्रिया प्रदर्श्यते। तद्यथा। छायाकर्णप्रमाणं यावत्तावत् १। अस्माद्भुजः साध्यः। त्रिभज्याहृतार्काग्रका कर्णनिघ्नीत्यादिना दक्षिणगोल उत्तरा जाता कर्णवृत्ताग्रा।

atropapattirbījagaṇitaprakriyayā tatrāvyaktaṃ yākāropalakṣitaṃ trijyā-grādikā ādyākṣaropalakṣitāḥ kṛtvā bījaprakriyā pradarśyate | tadyathā chāyākarṇapramāṇaṃ yāvattāvat 1 | asmādbhujaḥ sādhyaḥ | tribhajyāhṛtārkāgrakā karṇanighnītyādinā dakṣiṇagola uttarā jātā karṇavṛttāgrā |

The *upapatti* is through the process of algebra. The algebraic process is demonstrated by giving some symbol for the unknown and the first letters are chosen for Radius, *agrā* and so on. This is how it is. Let the unknown *chāyākarṇa* be denoted by *yāvattāvat* (K). From this the *bhujā* (B) is to be found out. In the southern hemisphere, the *karṇāgrā* is equal to the product of *karṇa* and *agrā* divided by the radius, and is considered northwards (*uttarājātā*).

This *karṇāgrā* is symbolically denoted in Sanskrit texts as follows. Here the alphabets या and अ denote the variables *Karṇa* (K) and *agrā* ($\mathcal{A}$) respectively. The number 1 by the side of अ denotes the coefficient of that variable.

या . अ १
त्रि १ $$\frac{K\times\mathcal{A}}{R}.$$

Bhāskara considers the southern hemisphere (southern declinations) first. He discusses the northern declinations later.

इयं कर्णवृत्ताग्रा पलच्छायया संस्कृता जातो भुजः ।
iyaṃ karṇavṛttāgrā palacchāyayā saṃskṛtā jāto bhujaḥ |

या . अ १ वि
त्रि १ $$\frac{K\times\mathcal{A}}{R}+s.$$

अस्मात् त्रिज्याहतोऽसौ प्रभया विभक्त इत्यादिना दिग्ज्या साध्या ।
asmāt trijyāhato'sau prabhayā vibhakta ityādinā digjyā sādhyā |

This *agrā* on the *karṇa* circle corrected by the equinoctial shadow is the *bhuja*. Multiplying this (B) by *trijyā* (R) and dividing by the shadow (S), the *digjyā* is to be obtained.

Here the positive sign for the correction to the *karṇavṛttāgrā* is for southern declinations.

अयं त्रिज्यागुणितः या. अ १ वि. त्रि १। कर्णवर्गाद्द्वादशवर्गेऽपनीते जातश्छायावर्गः याव १ रू १ ४ ४। वर्गेण वर्गं गुणयेद्भजेच्चेत्यनेन पूर्वराशिवर्गो भाज्यः। पूर्वराशेर्यावद्वर्गः क्रियते तावत् प्रथमं यावत्तवद्वर्गगुणितोऽग्रावर्गः। ततोयाकारगुणितोऽग्रात्रिज्यापलभानां घातो द्विगुणस्ततः पलभावर्गगुणस्त्रिज्यावर्गो रूपराशिरन्ते भवति। स तेन छायावर्गेण भक्तो जातः –

ayaṃ trijyāguṇitaḥ yā. a 1 vi. tri 1 | karṇavargāddvādaśavarge'panīte jātaśchāyāvargaḥ yāva 1 rū 1 4 4 | vargeṇa vargaṃ guṇayedbhajeccetyanena pūrvarāśivargo bhājyaḥ pūrvarāśeryāvadvargaḥ kriyate tāvat prathamaṃ yāvattavadvargaguṇito'grāvargaḥ tatoyākāraguṇito'grātrijyāpalabhānāṃ ghāto dviguṇastataḥ palabhāvargaguṇastrijyāvargo rūparāśirante bhavati sa tena chāyāvargeṇa bhakto jātaḥ –

This is multiplied by the *trijyā* (R): $K\times\mathcal{A}+s\times R$. When the square of 12 is subtracted from the square of *karṇa* (K), the result is the square of the shadow: $K^2-12^2=S^2$. Since it is said that 'one should multiply or divide a square by a

square' [BīGa2009, p. 16], the square of the previous quantity (that is, $K\times\mathcal{A}+s\times R$) is divided by this ($S^2$). When the previous quantity is squared, the first term is the square of *agrā* multiplied by the square of *yāvat tāvat* (K), then twice the product of *agrā* ($\mathcal{A}$), Radius (*trijyā*), and *palabhā* multiplied by the *yā*, then at the end, the square of Radius (*trijyā*) multiplied by the square of *palabhā*. When these are divided by the square of the shadow, we have

याव.अव १ या.अ.वि.त्रि २ विव.त्रिव १
याव १ रु १४̇४

$$\frac{K^2\times\mathcal{A}^2+2\times K\times\mathcal{A}\times s\times R+s^2\times R^2}{K^2-144}.$$

The text proceeds further to describe the algebraic manipulation to be done as follows: The algebraic expression denoted above symbolically is the square of *digjyā* (D^2). That is,

$$D^2=\frac{K^2\times\mathcal{A}^2\ldots\ldots\times R^2}{K^2-144}.$$

अत्र फलं दिग्ज्यावर्गः| अतोऽयं दिग्ज्यावर्गेण समः क्रियते। अत्र पक्षौ समच्छेदीकृत्य छेदगमे तयोः शोधनार्थं न्यासः ।

atra phalaṃ digjyāvargaḥ ato'yaṃ digjyāvargeṇa samaḥ kriyate | atra pakṣau samacchedīkṛtya chedagame tayoḥ śodhanārthaṃ nyāsaḥ |

This is equated to the square of *digjyā*. Having reduced both sides to equal denominators and removed the denominators, one places them for the equal subtraction.

याव.अव १ या.अ.वि.त्रि २विव.त्रि १
याव दिव १ या ० दिव १४̇४

$$\frac{K^2\mathcal{A}^2+2\times K\times\mathcal{A}\times R\times s+s^2\times R^2}{K^2\times D^2+0K-144\times D^2}.$$

अत्रैकाव्यक्तं शोधयेदन्यपक्षादित्यादिना समशोधने कृते जातं प्रथमपक्षे प्रथमस्थाने दिग्ज्याग्रावर्गान्तरं यावद्वर्गगुणितं द्वितीयस्थाने त्रिज्याक्षभाग्राभिहतिर्द्विगुणिता यावत्तावद्गुणिता ऋणगता च। द्वितीयपक्षे रूपस्थाने व्यासार्धवर्गः पलभाकृतिघ्नो दिग्ज्याकृतिर्द्वादशवर्गनिघ्नी तत्संयुतिर्जाता। शोधितपक्षयोर्न्यासः।

atraikāvyaktaṃ śodhayedanyapakṣādityādinā samaśodhane kṛte jātaṃ prathamapakṣe prathamasthāne digjyāgrāvargāntaraṃ yāvadvargaguṇitaṃ dvitīyasthāne trijyākṣabhāgrābhihatirdviguṇitā yāvattāvadguṇitā ṛṇagatā ca dvitīyapakṣe rūpasthāne vyāsārdhavargaḥ palabhākṛtighno digjyākṛtirdvādaśavarganighnī tatsaṃyutirjātā | śodhitapakṣayornyāsaḥ |

When the equal subtraction is made according to (the rule), 'one should subtract the unknown of one (side) from the other side' [BīGa2009, p. 40], what is generated is: In the first (left) side, at the first position (we have) the difference of the squares of *digjyā* (D) and *agrā* ($\mathcal{A}$) multiplied by the square of *yāvat*; at the second position, negative of the product of *trijyā* (R), *akṣabhā* (s), and *agrā* multiplied by two and *yāvattāvat*, [that is, $K^2(D^2-\mathcal{A}^2)-2\times K\times s\times R\times\mathcal{A}$]. In the second (right) side, at the position of the constant (*rūpa*), the sum resulting from the product of the squares of *palabhā* (s) and *trijyā* (R) and the product of the squares of *digjyā* and 12, [that is, $s^2R^2+D^212^2$]. The two sides corrected (thus) are equated.

याव.दिव १ याव.अव१ या.अ.वि.त्रि २ विव.त्रिव १ दिव १४४ $\dfrac{K^2D^2 - K^2\mathcal{A}^2 - 2 \times K \times \mathcal{A} \times R \times s}{s^2R^2 + D^2 144}$

अथ पक्षयोर्मूलार्थं दिग्ज्याग्रावर्गवियोगेनापवर्तनं कृतम्। अव्यक्तवर्गस्थाने रूपं जातम्। इतरौ राशी अपवर्त्तितौ जातौ लघू। तत्र यो रूपराशिः सोऽत्र प्रथमसंज्ञः कृतः। अव्यक्तस्थाने त्रिज्याक्षभाग्राभिहतिर्दिग्ज्याग्रावर्गवियोगभक्ता चान्यसंज्ञः कृतः। इदानीं पक्षयोरन्यवर्गतुल्यानि रूपाणि प्रक्षिप्याव्यक्तपक्षस्य मूलम्। या १ अन्यः १। इदं प्रथमपक्षमूलम्। अथान्यवर्गेण युताद्यराशेर्मूलं द्वितीयपक्षमूलम्। तेन सह प्रथमपक्षमूलस्य पुनः समीकरणम्। तत्र प्रथमपक्षमूले यो अन्यो रूपराशिः स द्वितीयपक्षमूले समशोधने ऋणगतत्वात् क्षेप्यो भवति दक्षिणगोले। उत्तरगोले तु धनगतत्वाच्छोध्यः।

atha pakṣayormūlārthaṃ digjyāgrāvargaviyogenāpavartanaṃ kṛtam avyaktavargasthāne rūpaṃ jātam itarau rāśī apavarttitau jātau laghū tatra yo rūparāśiḥ so'tra prathamasaṃjñaḥ kṛtaḥ avyaktasthāne trijyākṣabhāgrābhihatirdigjyāgrāvargaviyogabhaktā cānyasaṃjñaḥ kṛtaḥ idānīṃ pakṣayoranyavargatulyāni rūpāṇi prakṣipyāvyaktapakṣasya mūlam yā 1 anyaḥ 1 | idaṃ prathamapakṣamūlam | athānyavargeṇa yutādyarāśermūlaṃ dvitīyapakṣamūlam tena saha prathamapakṣamūlasya punaḥ samīkaraṇam | tatra prathamapakṣamūle yo.anyo rūparāśiḥ sa dvitīyapakṣamūle samaśodhane ṛṇagatatvāt kṣepyo bhavati dakṣiṇagole | uttaragole tu dhanagatatvācchodhyaḥ |

In order to take the square roots of both sides, reduction (division) by the unknown K, all the terms have to be divided by the difference of the squares of *digjyā* and *agrā* ($D^2 - \mathcal{A}^2$). [By doing so], at the place of the square of the unknown becomes unity is produced. The other two quantities reduced [by $D^2 - \mathcal{A}^2$] becomes smaller. Here whatever is the constant term *ruparāśi* that is called *prathama.* At the place of the unknown, the product of the radius, equinoctial shadow, and *agrā* divided by the difference of the squares of the *digjyā* and *agrā* ($\frac{ARs}{D^2 \sim \mathcal{A}^2}$) is called *anya.* Let the constant quantity equal to the square of *anya* be added to both sides: the square root of the unknown side is *yā* (K) - *anyā*: this is the square root of the first [left] side. The square root of the *ādya* to which the square of *anya* is added, is the square root of the second (right) side. This is again equated to the root of the first side. The 'constant' term in the root of the first side is to be added to the root of the second side in the process of the equal subtraction because it is negative in the southern hemisphere. In the northern hemisphere, it should be subtracted [from the root of the second side], as it appears as a positive [in the first side].

Here it should be noted that in the northern hemisphere (δ north), $B.R = S.R \sim K.\mathcal{A}$ from equation (42) . The quadratic equation in this case is

$$K^2 + \frac{2 \times K \times \mathcal{A} \times R \times s}{D^2 - \mathcal{A}^2} = \frac{s^2R^2 + D^2 144}{D^2 - \mathcal{A}^2}.$$

यदा उत्तरगोलेऽग्राया अल्पे दिग्गुण इच्छादिक्छायासाधनं तदा दिग्ज्यावर्गादग्रावर्गो न शुध्यति। अतः समक्रियायां विलोमशोधने क्रियमाणेऽव्यक्तपक्षमूलेऽन्य ऋणगतो लभ्यते स च द्वितीयपक्षमूलादधिकः स्यात् तदा

अव्यक्तमूलर्णगरूपतोऽल्पं व्यक्तस्य पक्षस्य पदं यदि स्यात् ।
ऋणं धनं तच्च विधाय साध्यमव्यक्तमानं द्विविधं क्वचित् तत् ॥

इत्यस्याः परिभाषायाः विषयः। अतस्तत्र द्विधा श्रुतिः स्यादित्युपपन्नम् ।

yada uttaragole'grāyā alpe digguṇa icchādikchāyāsādhanaṃ tadā digjyāvargādagrāvargo na śudhyati ataḥ samakriyāyāṃ vilomaśodhane kriyamāṇe'vyaktapakṣamūle'nya ṛṇagato labhyate sa ca dvitīyapakṣamūlādadhikaḥ syāt tadā

> *avyaktamūlarṇagarūpato'lpaṃ vyaktasya pakṣasya padaṃ yadi syāt|*
> *ṛṇaṃ dhanaṃ tacca vidhāya sādhyamavyaktamānaṃ dvividhaṃ kvacit tat ||*

ityasyāḥ paribhāṣāyāḥ viṣayaḥ | atastatra dvidhā śrutiḥ syādityupapannam |

When in the northern hemisphere, one determines the shadow of any optional direction for *digjyā* which is less than *agrā*, the square of *agrā* cannot be subtracted from the square of *digjyā*. Therefore, in the equation, the reverse subtraction being made, if the *anya* in the unknown side is obtained as a negative (quantity) and is greater than the square root of the second side, then it is an object of this metarule (*paribhāṣā*) [BīGa2009].

> If the square root of the known side of the quadratic be less than the negative absolute term occurring in the square root of the unknown side, then making it negative as well as positive, sometimes, two values of the unknown may be possible.

Thus it is proved that the (shadow) hypotenuse has two values.

[Translation by Sita Sundar Ram]

In the *upapatti*, *prathama* or *ādya* is $|x|$, and *anya* is $|y|$, where x and y are defined in (46). Bhāskara notes that the *anya* term comes with a negative sign in the LHS, when the *digjyā* is less than the *agrā*, or $D < \mathcal{A}$, and the square root of the left side is $K-$ *anya*, then. The square root of the right side would be numerically less than *anya*. Hence, two roots are possible in this case. As we have explained earlier, *digjyā* can be less than the *agrā* only, for northern declinations. In the *upapatti* also, the existence of two roots is discussed in the context of the “northern hemisphere” only.

5 Concluding remarks

In the beginning of *Grahagaṇita*, in verse 4 of *Kālamānādhyāya* of *Madhyamādhikāra*, Bhāskara says [SiŚi2005, p. 2]:

कृता यद्यप्याद्यैश्चतुरवचना ग्रन्थरचना
तथाप्यारब्धेयं तदुदितविशेषान् निगदितुम् ।
मया मध्ये मध्ये त इह हि यथास्थाननिहिता
विलोक्याऽतः कृत्स्ना सुजनगणकैर्मत्कृतिरपि ॥ ४ ॥

kṛtā yadyapyādyaiścaturavacanā grantharacanā
tathāpyārabdheyaṃ taduditaviśeṣān nigaditum
mayā madhye madhye ta iha hi yathāsthānanihitā
vilokyā'taḥ kṛtsnā sujanagaṇakairmatkṛtirapi ||4||

Ancient astronomers did write, of course, works abounding in intelligent expression; nonetheless, **this work is started by me to give better expression to [or improve] some of their special/important statements. These [improvements] are given by me here and there in their respective places.** So, I beseech the good-minded mathematicians to go through this entire work of mine also.

[Translation by Arkasomayaji, revised by the author.]

Bhāskara lives up to his promise. In this work, most of the standard calculations and algorithms in Indian astronomy of his times are included, mistakes in many of them are rectified, generalisations are made where necessary, and many new results are presented. All these are presented in the source verses of the text, and are explained in detail in his own commentary, *Vāsanābhāṣya*. In this paper, we have discussed some representative topics in the work, which would throw light on the methodology of the eminent astronomer-mathematician.

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Bhāskarācārya and *Kṣayamāsa*

Michio Yano*

1 Introduction

One of the important topics of the traditional luni-solar calendar is how to insert intercalary months so that the lunar month, which is dependent on the lunar phase, can be adjusted to the solar month, which is determined by the position of the sun. The period of intercalations was studied by ancient astronomers and calendar makers. In the *Vedāṅgajyotiṣa*,[1] the oldest Indian text on astronomy and the calendar, a very crude period of five years (called *pañcābdayuga*) was used, where two intercalary months were inserted in every five years. If such a crude calendar was in practice, one could have easily found the discrepancy between theory and practice (or observation). For instance, a new moon would not be observed even if the *tithi* was the second or sometimes the third from the conjunction according to a calendar theoretically prepared. Later the better cycle of nineteen years was known in India as well as in Greece (the so-called Metonic Cycle) and China.

What characterizes the Indian calendar is the practice of the omitted month (*kṣayamāsa*), which is the opposite case of an intercalary month: a lunar month is omitted from the naming. This is the custom that is found uniquely in India. In China and Japan, the possibility of the occurrence of the omitted month was known around the beginning of the nineteenth century, but it was not put in practice. The practice of the omitted month started, of course, as a result of the advancement of theory. But such an advanced theory is not found in the

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[1] We have two recensions of the *Vedāṅgajyotiṣa*, one belonging to the *Ṛgveda*, the other to the *Yajurveda*, but they are transmitted only in very corrupt form.

K. Ramasubramanian et al. (eds.), *Bhāskara-prabhā*, Sources and Studies in the History of Mathematics and Physical Sciences, https://doi.org/10.1007/978-981-13-6034-3_12

earlier Sanskrit texts. As we will see shortly the first text which clearly states the possibility of the omitted month is the *Grahagaṇitādhyāya* of Bhāskara's[2] *Siddāntaśiromaṇi*.

Indian month names: In order to adjust the lunar month to the solar month, the Indian people devised a clever method of naming the month. The solar month is determined by the sun's entry (*saṅkrānti*) into a new zodiacal sign[3] while the name of the lunar month is determined by the *saṅkrānti* which it contains. For instance, the synodic month which contains *meṣasaṅkrānti* (sun's entry into *Meṣa*) is called *Caitra*, because the full moon is seen against the background of the *nakṣatra Citrā* (Spica, α Virgo). The next month which contains *vṛṣasaṅkrānti* is called *Vaiśākha*, etc. Table 1 shows this correspondence.[4]

***Adhimāsa* or intercalary month:** Sometimes it happens that a lunar month contains no *saṅkrānti*, and thus one can not name it by the *saṅkrānti*. In such a case, the month is called by the name of the next month with the prefix *adhika-* ('additional'). In Figure 1, the month after *Vaiśākha* has no *saṅkrānti* and it is named *Adhika-Jyaiṣṭha*. Roughly speaking the intercalary month (*adhimāsa* or *adhikamāsa*) occurs every 2.7 years (or 7 intercalary months in 19 years).

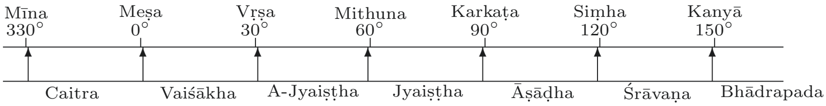

Figure 1: Example of *adhimāsa*.

[2] For the sake of brevity I just say Bhāskara without *-ācārya*. Of course, I admire him as my *ācārya*.

[3] The concept of the zodiacal signs as the reference system of the ecliptic coordinates was transmitted to India from the west sometime in the early centuries of the Christian era. Before the zodiacal signs were known, ancient Indian people used the derivative name of the lunar mansion (*nakṣatra*) where the full moon stays. Thus, for example, the solar *Meṣa* month in Table 1 was called *saura-caitra-māsa* or solar *Caitra* month.

[4] A similar scheme is found in the Chinese and Japanese luni-solar calendar. A solar year is divided into 24 parts by *jié qì* 'solar term', out of which 12 are called *zhōng qì* 'center term' which corresponds to Indian *saṅkrānti*. There is a significant difference between Indian and Chinese calendrical systems. While Chinese ecliptic coordinates are tropical, Indian coordinates are sidereal, namely, the beginning of the ecliptic coordinates are fixed without the precession of equinoxes taken into account.

Table 1: *Saṅkrānti* and month name.

Saṅkrānti	degrees	month
Meṣa	0	*Caitra*
Vṛṣa	30	*Vaiśākha*
Mithuna	60	*Jyaiṣṭha*
Karkaṭa	90	*Āṣāḍha*
Siṃha	120	*Śrāvaṇa*
Kanyā	150	*Bhādrapada*
Tulā	180	*Āśvina*
Vṛścika	210	*Kārttika*
Dhanus	240	*Mārgaśīrṣa*
Makara	270	*Pauṣa*
Kumbha	300	*Māgha*
Mīna	330	*Phālguna*

***Kṣayamāsa* or omitted month:** If the mean motions of the sun and moon are used in this calendar computation, there is no possibility for a synodic months to contain two *saṅkrāntis*.

With the advancement in theoretical formulations of astronomy, the exact time of the *saṅkrānti* was determined by the true motion of the sun and the true synodic month was used. Thus, although very rarely, it happens that a synodic month may contain two *saṅkrāntis*. In this case, the first *saṅkrānti* is used for the naming of the month, and the lunar month name corresponding to the second *saṅkrānti* is omitted. This is the so-called *kṣayamāsa*, namely, omitted month.

Figure 2 illustrates the case of the *kṣayamāsa* which, theoretically,[5] should have occurred in the year *Śaka* 974 (expired), or 1052-3 CE.[6] In this year two *saṅkrāntis*, namely, *dhanus-saṅkrānti* and *makara-saṅkrānti*, occurred in one synodic month (*Mārgaśīrṣa*). Thus *Pauṣa* month which is defined as the month containing the *makara-saṅkrānti* is omitted and the next month is called *Māgha* which is defined by the *kumbhasaṅkrānti*. Consequently there is no month called *Pauṣa* in this year. Usually in such a year two intercalary

[5] I used the word ‘theoretically’ with very significant meaning, of course. I wish I could find a *pañcāṅga* of *Śaka* 974!

[6] I follow the *amānta* (new moon ending) system of naming the lunar months instead of the *pūrṇimānta* (full moon ending) system.

months occur shortly before and after the *kṣayamāsa*. Therefore, the total number of months results in 13. In this example *Adhika-Aśvina* and *Adhika-Caitra* were inserted.

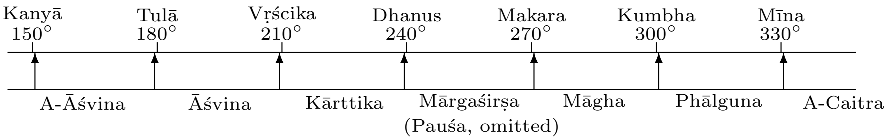

Figure 2: Example of *kṣayamāsa*.

2 *Kṣayamāsa* and the frequency of its occurence

As far as I know, the first Indian astronomer who discussed in detail the problem of the *kṣayamāsa* is Bhāskara. Arkasomayaji [SiŚi1980, p. 74] says that a mention of the occurrence of *kṣayamāsa* was made by Śrīpati (fl. *ca.* 1050 CE) first, without giving any textual evidence. Probably he followed Sewell and Dikshit [SD1896, p. 28]. Śrīpati's *Siddhāntaśekhara* was not available to Sewell and Dikshit, but they quote two verses ascribed to the *Siddhāntaśekhara* which are cited in a *muhūrta* work called *Jyotiṣa-darpaṇa* (dated 1557 CE according to [SD1896, p. 27]):

> मध्यमरविसङ्क्रान्तिप्रवेशरहितो भवेद् अधिकः
> मध्यश्चान्द्रोमासो मध्याधिकलक्षणं चैतत् ।
> विद्वांसस्त्वाचार्या निरस्य मध्याधिकं मासं
> कुर्युः स्फुटमासेन हि यतोऽधिकः स्पष्ट एव स्यात्॥
>
> *madhyamaravisaṅkrāntipraveśarahito bhaved adhikaḥ*
> *madhyaś cāndro māso madhyādhikalakṣaṇaṃ caitat* |
> *vidvāṃsas tv ācāryā nirasya madhyādhikaṃ māsaṃ*
> *kuryuḥ sphuṭamāsena hi yato'dhikaḥ spaṣṭa eva syāt* ||
>
> The lunar month which has no mean sun's entrance into a sign shall be a mean intercalary month. This is the definition of a mean added month. The learned *Ācāryas* should leave [using] the mean added month, and should go by apparent reckoning, by which the added month would be apparent (true).
>
> (Translation by Sewell-Dikshit)

The use of true motions of the sun and moon is stressed here, but I can not find these verses in the published text of the *Siddhāntaśekhara*.

The first explicit reference to *kṣayamāsa* is found in Bhāskara's *Siddhāntaśiromaṇi, Grahagaṇitādhyāya, Madhyamādhikāra, Adhimāsāvamanirṇayādhyāya*.[7] First he gives the definition [SiŚi1939, p. 69]:

असङ्क्रान्तिमासोऽधिमासः स्फुटः स्यात्
द्विसङ्क्रान्तिमासः क्षयाख्यः कदाचित्।
क्षयः कार्त्तिकादित्रये नान्यतः स्यात्
तदा वर्षमध्येऽधिमासद्वयं च॥ ६ ॥

asaṅkrāntimāso'dhimāsaḥ sphuṭaḥ syāt
dvisaṅkrāntimāsaḥ kṣayākhyaḥ kadācit |
kṣayaḥ kārttikāditraye nānyataḥ syāt
tadā varṣamadhye'dhimāsadvayaṃ ca || 6 ||

The (lunar) month which has no *saṅkrānti* would be the true additional month. Sometimes there is a month having two *saṅkrāntis*, which is called 'omitted' (*kṣaya). Kṣaya* would occur in the three months beginning with *Kārttika*, and not otherwise. Then there are two *adhimāsas* within a year.

In his auto-commentary Bhāskara further explains:

यस्मिन् शशिमासेऽर्कसंङ्क्रान्तिर्नास्ति सोऽधिमास इति प्रसिद्धम् । तथा यत्र मासे सङ्क्रान्तिद्वयं भवति स क्षयमासो ज्ञेयः । यतः सङ्क्रान्त्युपलक्षिता मासाः । अत एकस्मिन् मासे सङ्क्रान्तिद्वये जाते सति मासयुगुलं जातम् । स क्षयमासः । कदाचित् कालान्तरे भवति । यदा भवति तदा कार्तिकादित्रय एव । तदा क्षयमासात् पूर्वमासत्रयान्तर एकोऽधिमासोऽग्रतश्च मासत्रयान्तरितोऽन्यश्चासङ्क्रान्तिमासः स्यात् ।

अत्रोपपत्तिः । चन्द्रमासप्रमाणम् एकोनत्रिंशत् सावनदिनान्येकत्रिंशद् घटिकाः पञ्चाशत् पलानि २९ । ३१ । ५० तथार्कमासस्त्रिंशद्दिनानि षड्विंशतिर्घटिकाः सप्तदश पलानि ३० । २६ । १७ एतावद्भिर्दिवसै रविर्मध्यमगत्या राशिं गच्छति । यदार्कगतिरेकषष्टिः कलास्तदा सार्धैकोनत्रिंशता दिनैः २९ । ३० राशिं गच्छति । अतश्चान्द्रमासादाल्पोऽर्कमासस्तदा स्यात् । एवं रविमासस्य परमाल्पता २९ । २० । ४८ सा चैकषष्टिर्गतिर्वृश्चिकादित्रयेऽर्कस्य । स ईदृशोऽल्पोऽर्कमासो यदा चान्द्रमासस्यानल्पस्यान्तःपाती भवति तदैकस्मिन् मासे रविसङ्क्रमणद्वयं उपपद्यते । अत उक्तं -- क्षयः कार्तिकादित्रय इति । पूर्वं किल भाद्रपदोऽसङ्क्रान्तिर्जातस्ततोऽर्कगतेरधिकत्वान्मार्गशीर्षो द्विसङ्क्रान्तिः । ततः पुनर्गतेरल्पत्वाच्चैत्रोऽप्यसङ्क्रान्तिर्भवति । ततो वर्षमध्येऽधिमासद्वयं इत्युपपन्नम् ।

yasmin śaśimāse'rkasaṅkrāntir nāsti so'dhimāsa iti prasiddham | *tathā yatra māse saṅkrāntidvayaṃ bhavati sa kṣayamāso jñeyaḥ* | *yataḥ saṅkrāntyupalakṣitā māsāḥ* | *ata ekasmin māse saṅkrāntidvaye jāte sati māsayugulaṃ jātam* | *sa kṣayamāsaḥ* | *kadācit kālāntare bhavati* | *yadā bhavati tadā kārtikāditraya eva* | *tadā kṣayamāsāt pūrvamāsātrayāntara eko 'dhimāso 'grataś ca māsatrayāntarito 'nyaś cāsaṅkrāntimāsaḥ syāt* |

atropapattiḥ | *candramāsapramāṇam ekonatriṃśat sāvanadināny ekatriṃśad ghaṭikāḥ pañcāśat palāni 29* | *31* | *50 tathārkamāsas triṃśaddināni ṣaḍviṃśatir*

[7] There are many different editions of the *Grahagaṇitādhyāya*. I have used the Ānandāśrama edition (No. 110 in two parts) edited by Vināyaka Gaṇeśa Āpaṭe, 1939 [SiŚi1939].

ghaṭikāḥ saptadaśa palāni 30 | 26 | 17 etāvadbhir divasai ravir madhyamagatyā rāśiṃ gacchati | yadārkagatir ekaṣaṣṭiḥ kalās tadā sārdhaikonatriṃśatā dinaiḥ 29 | 30 rāśiṃ gacchati | ataś cāndramāsād alpo 'rkamāsas tadā syāt | evaṃ ravimāsasya paramālpatā 29 | 20 | 48 sā caikaṣaṣṭir gatir vṛścikāditraye 'rkasya | sa īdṛśo 'lpo 'rkamāso yadā cāndramāsasyānalpasyāntaḥpātī bhavati tadaikasmin māse ravisaṅkramaṇadvayam upapadyate | ata uktaṃ—kṣayaḥ kārtikāditraya iti | pūrvaṃ kila bhādrapado 'saṅkrāntir jātas tato 'rkagater adhikatvān mārgaśīrṣo dvisaṅkrāntiḥ | tataḥ punar gater alpatvāc caitro 'py asaṅkrāntir bhavati | tato varṣamadhye 'dhimāsadvayam ity upapannam |

It is clear that if there is no sun's *saṅkrānti* in a certain lunar month the month is *adhimāsa.* Likewise if there are two *saṅkrāntis* in a month, the month is to be known as *kṣayamāsa.* This is because the [lunar] month is defined by *saṅkrānti.* Therefore, if two *saṅkrāntis* are born in a month, two months are born. This is *kṣayamāsa.* Sometimes it occurs [after a large] interval of time. When it occurs, it is only in the three months beginning with *Kārttika.* In that time, in the three months before the *kṣayamāsa* one *adhimāsa* [would occur] and after it too, at the interval of three months, there would be another month having no *saṅkrānti.*

Here is the reasoning. The length of a [mean] lunar month is 29 civil days, 31 *ghaṭikās*, 50 *palas* (29;31,50). Likewise a [mean] solar month is 30 [civil] days 26 *ghaṭikās*, 17 *palas* (30;26,17). The sun traverses one sign [*rāśi*] with mean motion in this amount of days. When the motion of the sun is 61 *kalās* (i.e. minutes), then it traverses one sign in 29 and half days (29; 30). Therefore in this case the solar month is shorter than the lunar month. Likewise the shortest of the solar month is 29;20,48 (days). This 61 (*kalā*) motion (per day) of the sun is when it is in the three signs beginning with *Vṛścika* (Scorpion). If such a short solar month falls within a long lunar month, then two solar *saṅkrāntis* occur in one [lunar] month. Therefore it is said that '*Kṣaya* would be in the three months beginning with *Kārttika*'. First (*pūrvam*), perhaps[8] *Bhādrapada* without *saṅkrānti* is born and, after that, due to the swiftness of the solar motion, the *Mārgaśīrṣa* month will have two *saṅkrāntis.* Then again due to the slowness of the solar motion, *Caitra* month too, will have no *saṅkrānti.* Therefore, it is proper that there are two *adhimāsas* within a year.

Then Bhāskara gives examples:

गतोऽब्ध्यद्रिनन्दै (९७४) र्मिते शाककाले
तिथीशै (१११५) र्भविष्यत्यथाङ्गाक्षसूर्यैः (१२५६)।
गजाद्र्यग्निभूभिः (१३७८) तथा प्रायशोऽयं
कुवेदेन्दु (१४१) वर्षैः क्वचिद्गोकुभि (१९) श्च॥ ७ ॥

gato'bdhyadrinandair (974) mite śākakāle
tithīśair (1115) bhaviṣyaty athāṅgākṣasūryaiḥ (1256) |
gajādryagnibhūbhiḥ (1378) tathā prāyaso'yam
kuvedendu (141) varṣaiḥ kvacid gokubhiś (19) ca || 7 ||

When the time of *Śaka* measured by 974 [years] had expired [there occurred a *kṣayamāsa*], and it will occur in [*Śaka*] 1115, 1256, and 1378. Thus this is mostly in [every] 141 years and sometimes in [every] 19 years.

[8] Bhāskara uses 'perhaps' (*kila*) here. It seems that he wanted to say that it is not only *Bhādrapada.* In fact, in the example of Fig. 2 above, the first *adhimāsa* occurred in *Āśvina.* The word *kila* is also found in the beginning of his commentary on the next verse.

Basic constants: Bhāskara's astronomical constants are those of the *Brāhmapakṣa*. The basic numbers in a *Kalpa*, listed in chapter 2, *Madhyamādhikāra* [SiŚi1939] are given in Table 2. From the values presented there, it can be seen that the mean length of a synodic month and a solar month are:

$$\frac{D}{M} \approx 29;31,50,$$

$$\text{and} \qquad \frac{D}{M_s} \approx 30;26,17$$

respectively.

Table 2: Astronomical parameters given by Bhāskara.

Parameter	Notation	Number	Ref.
Solar years	$R_\odot = Y$	4,320,000,000	ch. 2.1
Rotations of the moon	R_m	57,753,300,000	ch. 2.2
Rotations of the sky	R_s	1,582,236,450,000	ch. 2.7
Solar days	$Y \times 360$	1,555,200,000,000	ch. 2.8
Solar months	$M_s = Y \times 12$	51,840,000,000	ch. 2.11
Lunar (synodic) months	$M = R_m - R_\odot$	53,433,300,000	ch. 2.11
Tithis	$T = M \times 30$	1,602,999,000,000	ch. 2.8
Civil days	$D = R_s - R_\odot$	1,577,916,450,000	ch. 2.9
Adhimāsas	$A = M - M_s$	1,593,300,000	ch. 2.10
Avamas (omitted days)	$U = T - D$	25,082,550,000	ch. 2.10

The frequency of *adhimāsas* is:

$$\frac{Y}{A} = \frac{43,200}{15,933} = \frac{14,400}{5,311}.$$

Simpler ratios are derived by the convergence of the continued fraction:

$$\frac{14,400}{5,311} = 2 + \cfrac{1}{1 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{6 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{7 + \cfrac{1}{3 + \frac{1}{2}}}}}}}}}$$

The various convergents of this continued fraction are: $\frac{2}{1}, \frac{3}{1}, \frac{8}{3}, \frac{19}{7}, \frac{122}{45}, \frac{141}{52}, \ldots$

It is quite probable that Bhāskara got the interval of '19 years' and '141 years'. In this way although he does not explicitly mention it he obviously knew the interval of 122 years, which is the difference of *Śaka* 1256 and 1378, found in verse 7 just quoted above.

In his auto-commentary Bhāskara argues for the possibility of a *kṣayamāsa* using the concept of *śuddhi*, which he introduced in the preceding section [SiŚi1939, ch. 5.1–7]. As is shown in Figure 3[9] the *śuddhi* is the remainder of an intercalary month (*adhimāsaśeṣa*) expressed in *tithis*.

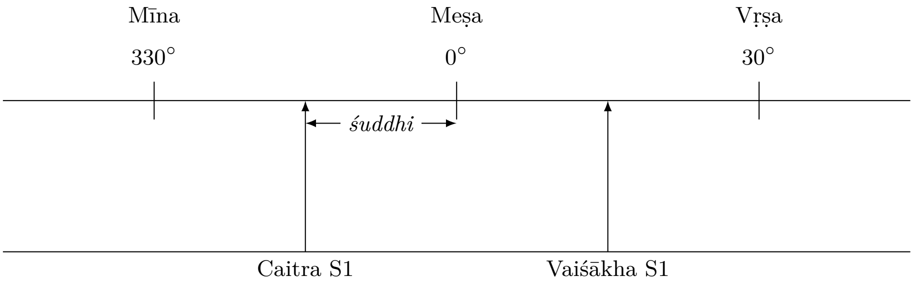

Figure 3: Occurence of *śuddhi*.

The average value of *śuddhi* per solar year is

$$\frac{1602999000000}{4320000000} - 360 = 11;3,52,30.$$

[9] In this figure S1, stands for *Śuklapakśa pratipad*, that is, the first *tithi* of the bright fortnight.

When exactly y solar years have elapsed since the epoch, the integer number of the elapsed *adhimāsas* ($[A]$) and the remainder (*śuddhi*) are related through the relation:

$$\frac{11;3,52,30}{30} \times y = [A]\frac{\acute{s}uddhi}{30}.$$

This is the value at the *meṣasaṅkrānti* of the $(y+1)^{\text{th}}$ year. This means that a similar amount of *śuddhi* will repeat at intervals of 19, 122, and 141 solar years.

Now let us learn from Bhāskara's own words:

यदा किलैकविंशतिः शुद्धिस्तदा भाद्रपदोऽधिमासः । तस्मिञ्जाते कार्तिकादित्रये क्षयमासः संभाव्यते । सा च तथाविधाशुद्धिः कुवेदेन्दु १४१ वर्षान्तरेकाले पुनर्भवति । किंतु सत्रिभागाभिः षड्भिर्घटिकाभिरधिका भवति । कदाचिदेकोनविंशत्या वर्षैस्तादृशी भवति । तत्र त्रिभागोनाभिश्चतुर्दशघटिकाभिरधिका भवति । कुवेदेन्दुवर्षेभ्यस्तथैकोनविंशतिवर्षेभ्यो 'द्विधाब्दा द्विरामैः खरामैश्च भक्ता' इत्यादिना लब्धेष्वधिमासेषु शेषतिथिषु शून्यं प्रथमस्थाने सत्र्यंशा षड्घटिकाः स्युः ६ । २० । द्वितीये वित्र्यंशाश्चतुर्दश १३ । ४० । अत उक्तं-प्रायशोऽयं कुवेदेन्दुवर्षैः क्वचिद् गोकुभिश्चेति । प्रागग्रतश्चेत्यर्थादुक्तं स्यात् ।

yadā kilaikaviṃśatiḥ śuddhis tadā bhādrapado'dhimāsaḥ | tasmiñ jāte kārtikāditraye kṣayamāsaḥ saṃbhāvyate | sā ca tathāvidhā śuddhiḥ kuvedendu 141 varṣāntare kāle punar bhavati | kiṃtu satribhāgābhiḥ ṣaḍbhir ghaṭikābhir adhikā bhavati | kadācid ekonaviṃśatyā varṣais tādṛśī bhavati | tatra tribhāgonābhiś caturdaśaghaṭikābhir adhikā bhavati | kuvedenduvarṣebhyas tathaikonaviṃśativarṣebhyo 'dvidhābdā dvirāmaiḥ kharāmaiś ca bhaktā' ity ādinā labdheṣvadhimāseṣu śeṣatithiṣu śūnyaṃ prathamasthāne satryaṃśā ṣaḍghaṭikāḥ syuḥ 6 | 20 | dvitīye vitryaṃśāś caturdaśa 13 | 40 | ata uktaṃ–prāyaśo'yaṃ kuvedenduvarṣaiḥ kvacid gokubhiś ceti | prāgagrataś cety arthād uktaṃ syāt |

When, perhaps, the *śuddhi* is 21, then *Bhādrapada* [would be] the *adhimāsa*. When it (*adhimāsa*) is born, then *kṣayamāsa* is made possible in the three months beginning with *Kārttika*. Such kind of *śuddhi* would occur again after the interval of 141 years. But it is longer by $6\frac{1}{3}$ *ghaṭikās*.[10] Sometime after the interval of 19 years there would be a similar [*śuddhi*]. In that case it is longer by 13;40 *ghaṭikās*.[11] At the interval of 141 years and 19 years, by [the rule] beginning with 'the years are divided in two ways, by 32 and by 30',[12] when the *adhimāsas* are obtained the remaining *tithis* are zero and [the remaining] *ghaṭikās* are 6;20 in the first case and 13;40 in the second case. Therefore it was said that 'mostly in every 141 years and sometimes 19 years'. As a matter of fact, 'before and after [19 years]' was intended.

With these two very succinct verses Bhāskara gives definition of the *kṣayamāsa*, the frequency of the phenomena, and the immediate past year in which the

[10] $11;3,52,30 \times 141 \approx 52$ months $+6;22$ *ghaṭikās*.

[11] $11;3,52,30 \times 19 \approx 7$ months $+13;40$ *ghaṭikās*.

[12] [SiŚi1939, ch. 5. 6] which gives an alternative method of computing *adhimāsa* and its remainder. *Adhimaāsas* ($[A]$) and *śuddhi* in y years $= 11y + \frac{y}{30} + \frac{y}{32} = [A] + \frac{\acute{s}uddhi}{30}$.

kṣayamāsa occured and the future years when it would occur. In order to check the validity of Bhāskara's words I listed the occurrence of *kṣayamāsa* (table 3) as they are found in the chronological table in [SD1896, p. 30]. A similar table is found in [Cha1998, p. 38].

This table agrees perfectly with the results of my *pañcāṅga* program which is based on the *Sūryasiddhānta*.[13] Even though the table begins with the year *Śaka* 326, when Indian astronomy was still in the primitive stage of development, there is no evidence that *kṣayamāsa* was in use at that time.

Those three years underlined in this table are mentioned by Bhāskara, but he skipped the four cases between *Śaka* 1115 and 1256, namely, 1180, 1199, 1218, 1237. Further, in *Śaka* 1378, which is mentioned by Bhāskara, there was no *kṣayamāsa*, according to [SD1896] and my program.

Such a difference may be due to the fact that Bhāskara's parameters for the sun and moon are slightly different from those of the *Sūryasiddhānta*. Although Bhāskara says that the possibility of the occurrence of *kṣayamāsa* is 'mostly' (*prāyaśas*) at the interval of 19 years and 141 years, we know from Table 3 that 38 years, 46 years, 65 years, 76 years, and 122 years are also possible [SD1896, p. 29].

Bhāskara says that the possible omitted months are *Kārttika, Mārgaśīrṣa*, and *Pauṣa*, but actually in later years *Māgha* is also possible as is shown in nos. 21 and 27 in Table 3.

Gaṇeśa's commentary: In the *Śiromaṇiprakāśa*, which is a commentary on the *Siddhāntaśiromaṇi*, Gaṇeśa (1600-1650) from Nandipura of Konkan (present village called Nāndgāon) [Dik1981, p. 128], quotes three verses from the work of Gaṇeśa Daivajña (b. 1507), the author of the *Grahalāghava*, which gives a list of *Śaka* years when *kṣayamāsa* occurred or would occur. They are:

> 1462, <u>1481</u>, 1603, 1744, <u>1763</u>, 1885, <u>1904</u>, 2026, 2045, <u>2129</u>, 2148, 2167, <u>2186</u>, 2232, <u>2251</u>, 2373, 2392,[14] 2514, 2533, 2655, 2674, 2796, 2815.

Those years which are underlined are given by Gaṇeśa Daivajña separately in the second half of the verses and Gaṇeśa comments that they are according to the *Āryapakṣa* (Āryabhaṭa's school).

Since Sewell-Dikshit's table is available only down to 1900 CE, I used my *pañcāṅga* program and confirmed that in those years excluding the underlined years there occurs *kṣayamāsa*. They are 2026 (*Pauṣa*), 2045 (*Māgha*), 2148 (*Pauṣa*), 2167 (*Pauṣa*), 2232 (*Pauṣa*), 2373 (*Pauṣa*), 2514 (*Pauṣa*), 2533

[13] This program was written by Dr. Makoto Fushimi and myself. It is open to the public and one can run it on my webpage: http://www.cc.kyoto-su.ac.jp/~yanom/pancanga/index.html.

[14] According to my program, this year contains an *adhimāsa* but no *kṣayamāsa*.

Table 3: Occurrence of *kṣayamāsas*.

Sl. No.	Year of occurrence		Name of the month	Interval
	Śaka	CE		(in years)
1	326	404	*Mārgaśīrṣa*	
2	345	423	*Pauṣa*	19
3	410	488	*Pauṣa*	65
4	448	526	*Pauṣa*	38
5	467	545	*Pauṣa*	19
6	486	564	*Pauṣa*	19
7	532	610	*Mārgaśīrṣa*	46
8	551	629	*Pauṣa*	19
9	692	770	*Pauṣa*	141
10	814	892	*Mārgaśīrṣa*	122
11	833	911	*Pauṣa*	19
12	<u>974</u>	1052	*Pauṣa*	141
13	<u>1115</u>	1193	*Pauṣa*	141
14	1180	1258	*Pauṣa*	65
15	1199	1277	*Pauṣa*	19
16	1218	1295	*Pauṣa*	19
17	1237	1315	*Mārgaśīrṣa*	19
18	<u>1256</u>	1334	*Pauṣa*	19
19	1302	1380	*Mārgaśīrṣa*	46
20	1321	1399	*Pauṣa*	19
21	1397	1475	*Māgha*	76
22	1443	1521	*Mārgaśīrṣa*	46
23	1462	1540	*Pauṣa*	19
24	1603	1681	*Pauṣa*	141
25	1744	1822	*Pauṣa*	141
26	1885	1963	*Pauṣa*	141
27	1904	1982	*Māgha*	19

(*Māgha*), 2655 (*Pauṣa*), 2674 (*Māgha*), 2796 (*Māgha*), 2815 (*Māgha*). From this we can guess that the occurrence of *kṣayamāsa* in *Māgha* is more frequent than in *Pauṣa* in later years.

As is shown in Table 3, the most recent past occurrence of the *kṣayamāsa* for us was *Śaka* year 1904, namely, 1982-83 CE. The year is not found in Gaṇeśa Daivajña's first list but he mentions it as a possible one according to the *Ārya-pakṣa*. Such non-agreement among the *pakṣas* (schools) should have caused some problems among the *pañcāṅga* makers. According to [Cha1998, p. 39], there are three different schools concerning how to deal with the *kṣayamāsa* and how to perform religious observances in two intercalary months and one omitted month. As I hear from a Japanese anthropologist, there was a big quarrel in West Bengal as to whether they should put two *adhimāsas* and omit one month or put only one *adhimāsa* in 1982–3 CE.

According to Gaṇeśa Daivajña's list and also according to my program, the next occurrence of *kṣayamāsa* will be in *Śaka* 2026 (= 2104 CE).[15]

3 Concluding remarks

Now let us think about the time when the theoretical problem of *kṣayamāsa* was first put into practice. Since the year *Śaka* 974 (1052 CE) was theoretically obtained by Bhāskara, he might have thought that the rare phenomenon was put in practice in this year. This is the time of Śrīpati, the author of the *Śiddhāntaśekhara*, who lived only about a half century before Bhāskara. As I mentioned above, Śrīpati might have known the possible occurrence of *kṣayamāsa*, but he did not give any explicit exposition of it. Neither did al-Bīrūnī (b. 976 - d. 1048 CE), a Persian contemporary of Śrīpati, mention this custom although he was very well versed in Indian astronomy and calendar making. Thus I think that it was shortly after Śrīpati and al-Bīrūnī that *kṣayamāsa* was put in practice and that Bhāskara was the first astronomer who clearly discussed this topic in Sanskrit.

The Report of the Calendar Reform Committee [RCRC1955, p. 246] admits that the occurrence of a *kṣaya* month was first known in India around 1100 CE without any textual evidence.

[15] Therefore we need not worry about a similar kind of problems while we are alive. It would be interesting to note that according to the traditional calendar in Japan and China we shall have a similar problem in 2033 CE. Some Japanese calendar makers are very much concerned about this problem, which is now called the '2033 problem' in Japan. You can search web pages of the internet with the key word 'Japan's year 2033 problem'.

Appendix

Reference to *kṣayamāsa* in a ritual text

I was first interested in this problem when I was asked by Prof. Yasuke Ikari, then professor of Kyoto University, about the reference to the *kṣayamāsa* in the *Garuḍapurāṇa*.[16] In chapter 13 of the *Sāroddhāra* which is a collection of the essential part of this *Purāṇa*, the following verses are found:

एक एव यदा मासः संक्रान्तिद्वयसंयुतः।
मासद्वयगतं श्राद्धं मलमासे हि शस्यते॥ १०० ॥
एकस्मिन् मासि मासौ द्वौ यदि स्यातां तयोर्द्वयोः।
तावेव पक्षौ ता एव तिथयस्त्रिंशदेव हि॥ १०१ ॥
तिथ्यर्धे प्रथमे पूर्वो द्वितीयेऽर्धे तदुत्तरः।
मासाविति बुधैश्चिन्त्यौ मलमासस्य मध्यगौ॥ १०२ ॥
असंक्रान्ते च कर्तव्यं सपिण्डीकरणं खग।
तथैव मासिकं श्राद्धं वार्षिकं प्रथमं तथा॥ १०३ ॥
संवत्सरश्च मध्ये तु यदि स्यादधिमासकः ।
तदा त्रयोदशे मासि क्रिया प्रेतस्य वार्षिकी॥ १०४ ॥
पिण्डवर्ज्यमसंक्रान्ते संक्रान्ते पिण्डसंयुतम्।
प्रतिसंवत्सरं श्राद्धम् एवं मासद्वयेऽपि च॥ १०५ ॥

eka eva yadā māsaḥ saṅkrāntidvayasaṃyutaḥ |
māsadvayagataṃ śrāddhaṃ malamāse hi śasyate || 100 ||
ekasmin māsi māsau dvau yadi syātāṃ tayor dvayoḥ |
tāv eva pakṣau tā eva tithayas triṃśad eva hi || 101 ||
tithyardhe prathame pūrvo dvitīye'rdhe taduttaraḥ |
māsāv iti budhaiś cintyau malamāsasya madhyagau || 102 ||
asaṅkrānte ca kartavyaṃ sapiṇḍīkaraṇaṃ khaga |
tathaiva māsikaṃ śrāddhaṃ vārṣikaṃ prathamaṃ tathā || 103 ||
saṃvatsaraś ca madhye tu yadi syād adhimāsakaḥ |
tadā trayodaśe māsi kriyā pretasya vārṣikī || 104 ||
piṇḍavarjyam asaṅkrānte saṅkrānte piṇḍasaṃyutam |
pratisaṃvatsaraṃ śrāddham evaṃ māsadvaye'pi ca || 105 ||

[GaPu1974, XIII, vv. 100–105]

When only one month is connected with two *saṅkrāntis*, the *śrāddha* which is prescribed in the two months is to be recommended in the *malamāsa*.

When there are two [solar] months in one [lunar] month, [then] there are only two half months [namely] 30 *tithis* in these two [solar months].

The wise should consider that in the first half of a *tithi* there is the first [month] and in the second half of it there is the next [month]. [Thus] two months should be considered as contained in the *malamāsa*.

[16] After I wrote a paper on it in 1989 in Japanese, I almost forgot it. Since I devised a *pañcāṅga* program in 1991 I got interested in the relation of theory and practice, but it was only recently that I began working again on the problem of *kṣayamāsa*.

> Oh bird (*Garuḍa*)! the *sapiṇḍa* (with rice-ball) ritual should be performed when there is no *saṅkrānti*. So too the monthly and first annual *śrāddha* rituals are to be performed.
>
> If there is a year having *adhimāsa* in the middle, the annual observance for the dead person should be performed in the thirteenth month.
>
> When there is no *saṅkrānti, piṇḍa* (rice-ball) is avoided. If there is *saṅkrānti, piṇḍa* is to be provided. Thus annual *śrāddha* should be performed in two months.

Usually *malamāsa* stands for an intercalary month, but in this context it seems to stand for *kṣayamāsa*, because it is used together with 'two *saṅkrāntis*'. In verses 103–105, with the word *asaṅkrānte*, the sacrificial ritual in the intercalary month is explained. Although the text is not very clear and verses 103 and 105 are contradictory,[17] this part of the *Garuḍapurāṇa* seems to belong to the time after the practice of omitting months started, namely, after *Śaka* 974. I would like to collect more ritual texts which deal with this topic.

Recently I met Mr. Hariprasad in a meeting called '*National Conference on Panchang Ganitham*' which was organized by Hindu Dharma Acharya Sabha in December 2010, at Tirumala. Mr. Hariprasad belongs to a *pañcaṅga* maker's family. He gave me a copy of a text entitled *Kṣayamāsakartavyanirṇaya*, i.e., 'Exposition of the observance in *kṣayamāsa*', written by his father Pidaparty Krishnamurty Sastri from Rajamahendravaram, Andhra Pradesh. The date is given as the full-moon day of *Śaka* 1883, Wednesday, which corresponds to March 21, 1962 CE. The text contains rich information concerning the history of the Indian calendar. Especially I am interested in the quotations from the *Kālamādhava*, ascribed to Mādhava (fl. ca.1360/1380).

Mr. Sastri should have been very conscious about the *kṣayamāsa* which was expected to occur next year. (See No. 27 of Table 2). I was impressed to know that the topic of *kṣayamāsa* continued to be one of the important ones among the *pañcāṅga* makers. I wish I could learn more from these traditional scholars.

[17] These verses were not properly understood by the modern translators.

Part V

The Siddhāntaśiromaṇi: Golādhyāya

सिद्धिं साध्यमुपैति यत्स्मरणतः क्षिप्रं प्रसादात्तथा
यस्याश्चित्रपदा स्वलङ्कृतिरलं लालित्यलीलावती ।
नृत्यन्ती मुखरङ्गगेव कृतिनां स्याद्भारती भारती
तं तां च प्रणिपत्य गोलममलं बालावबोधं ब्रुवे ॥

siddhiṃ sādhyam upaiti yatsmaraṇataḥ kṣipraṃ prasādāt tathā
yasyāś citrapadā svalaṅkṛtir alaṃ lālityalīlāvatī |
nṛtyantī mukharaṅgageva kṛtināṃ syād bhāratī bhāratī
taṃ tāṃ ca praṇipatya golam amalaṃ bālāvabodhaṃ bruve ||

Having paid obseisance to him (Gaṇeśa), by whose grace whatever is [desired] to be accomplished gets immediately accomplished, and likewise to her (Sarasvatī), by whose grace the speech of the [creative] writers dances on the stage of the tongue, filled with alluring words (*citrapadā*) various poetic embellishments (*svalaṅkṛti*) as well as playful elegance (*lālityalīlāvatī*)—like a well-adorned (*svalaṅkṛti*) dancer, dancing with grace and elegance (*lālityalīlāvatī*) with beautiful feet-work (*citrapadā*)—I state the spherics (*gola*) which is flawless (*amala*) and easily comprehensible [even] to uninitiated (*bāla*).

The *Vāsanābhāṣyas* of Bhāskarācārya

M. D. Srinivas*

1 Introduction

Apart from composing the celebrated textbooks of Indian mathematics and astronomy, viz. *Līlāvatī, Bījagaṇita* and *Siddhāntaśiromaṇi,* Bhāskarācārya also wrote the *Vāsanābhāṣyas,* commentaries which have acquired the status of canonical expository texts as they present detailed explanations and justifications for the results and processes outlined in these basic textbooks of mathematics and astronomy. From the various citations in the *Vāsanāvārttika* (c. 1621) of Nṛsiṃha Daivajña[1] on the *Vāsanābhāṣya* on the *Siddhāntaśiromaṇi,* it appears that Bhāskarācārya first composed the commentary, *Vivaraṇa,* [ŚiDh1981a] on the *Śiśyadhīvṛddhida* of Lallācārya. Apart from presenting *vivaraṇas* or explanations, this commentary also presents detailed *upapattis* (justifications, demonstrations or proofs). On the other hand, the

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[1] See [SiŚi1981]. Nṛsiṃha (b. 1586) was the grandson of Divākara of Golagrāma, a pupil of Gaṇeśa Daivajña, the famous author of the works *Grahalāghava* and the *Tithicintāmaṇi* and the *Buddhivilāsinī* commentary on the *Līlāvatī.* He received instruction from his uncles Viṣṇu and Mallāri (author of *Grahalāghavaṭīkā*). He wrote a commentary on the *Sūryasiddhānta* in 1611 and also a commentary on the *Tithicintāmaṇi* [CESS, Series A, vol. 3, (1976), pp. 204–206]. Nṛsiṃha's *Vāsanāvārttika,* written in Varanasi in 1621, is an extensive commentary on the *Vāsanābhāṣya* and serves to elucidate and clarify many of the issues discussed therein. As we shall see later, while discussing the planetary model of Bhāskara, there are indeed instances where Nṛsiṃha has not correctly grasped the import of Bhāskara's statements.

K. Ramasubramanian et al. (eds.), *Bhāskara-prabhā*, Sources and Studies in the History of Mathematics and Physical Sciences, https://doi.org/10.1007/978-981-13-6034-3_13

Vāsanā commentaries of Bhāskarācārya on the *Līlāvatī* and *Bījagaṇita* largely confine themselves to presenting the solutions of the examples (*uddeśakas*) presented in these texts. Of course, through such examples, we get to know the method of setting out the calculation (*nyāsa*) and the systematic way in which the calculation has to be carried out.

While commenting on the problems in the geometry section (*kṣetravyavahāra*) of the *Līlāvatī,* the *Vāsanā* makes extensive use of diagrams in order to explain the problem as well as the method of solution. There are also several instances where the *Līlāvatī-vāsanā* presents important explanations and insights. For example, the *vāsanā* on verse 164 of *Līlāvatī* [Līlā1937, p. 152] indicates how one can demonstrate the fact that a given a set of sides do not form a closed figure if any of the sides is less than or equal to the sum of the rest. The *vāsanā* on verse 168 of the *Līlāvatī,* which deals with the calculation of the base intercepts (*ābādhā*) of a triangle in which the foot of the altitude (*lamba*) falls outside the base, explains that here the computation leads to a negative value [Līlā1937, p. 155]:

ऋणगताऽऽबाधा दिग्वैपरीत्येन ।

ṛṇagatā"bādhā digvaiparītyena |

The intercept is negative, as it is oriented in the opposite direction.

The *vāsanā* on verse 175 of *Līlāvatī,* explains how to demonstrate, in the case of a given trapezium, that its area is correctly given by verse 173 as the product of half the altitude with the sum of the base and the face (*lambena kumukaikhyakhaṇḍam*); and not by the formula $[(s-a)(s-b)(s-c)(s-d)]^{\frac{1}{2}}$ involving s, the semi-perimeter (*sarvador-yutidalam*), mentioned as a rough value in verse 169 [Līlā1937, pp. 156, 162 and 165].

The *Bījagaṇita-vāsanā* is much more extensive and is literally filled with symbolic expressions which are employed in the representation and solution of equations. This commentary also offers several insightful explanations and discussions. For instance, the *vāsanā* on the very first example on the addition of positive and negative quantities, mentions that the unknown quantities (*avyakta*) are represented by their first letters and those that are negative have a dot placed above them. The verse 5 of *Bījagaṇita* defines a quantity divided by zero as *khahara* and, in his *vāsanā* dealing with the example of division of three by zero, *Bhāskara* remarks that this infinite quantity (*ananto rāśiḥ*) is called *khahara.* In the section on the elimination of the middle term, in a quadratic equation involving a single variable (*ekavarṇa-madhyamāharaṇa*), Bhāskara presents the problem of calculating the hypotenuse of a right angled triangle when the sides are given, and also asks for the demonstration (*upapatti*) of the well-known rule (the *bhujā-koṭi-karṇa-nyāya* or the so called Pythagoras theorem) employed in such a calculation. The *vāsanā* then outlines

two such demonstrations. Similarly, in the last section on equations containing product of unknowns (*bhāvita*), while discussing the solution of the equation $xy = 4x + 3y + 2$, the *vāsanā* again presents two demonstrations [BīGa2009, pp. 8, 11, 67, 68 and 104–106]. Here, Bhāskara also refers to the tradition of *upapatti* in Indian mathematics [BīGa2009, pp. 105–106]:

> अस्योपपत्तिः। सा च द्विधा सर्वत्र स्यात्। एका क्षेत्रगतान्या राशिगतेति। तत्र क्षेत्रगतोच्यते। ... अथ राशिगतोपपत्तिरुच्यते। सापि क्षेत्रमूलान्तर्भूता। ... इयमेव क्रिया पूर्वाचार्यैः संक्षिप्तपाठेन निबद्धा। ये क्षेत्रगताम् उपपत्तिं न बुध्यन्ति तेषाम् इयं राशिगता दर्शनीया।
>
> *asyopapattiḥ | sā ca dvidhā sarvatra syāt | ekā kṣetragatānyā rāśigateti | tatra kṣetragatocyate | …atha rāśigatopapattirucyate | sāpi kṣetramūlāntarbhūtā | …iyam eva kriyā pūrvācāryaiḥ saṃkṣiptapāṭhena nibaddhā | ye kṣetragatām upapattiṃ na budhyanti teṣām iyaṃ rāśigatā darśanīyā |*
>
> The demonstration of this: It is twofold in every case. One geometrical (*kṣetragatā*), and the other algebraic (based on quantities, *rāśigatā*). There, the geometrical one is stated. …Then the algebraic demonstration is stated. That (demonstration) is also geometry-based. …This procedure (of *upapatti*) has been earlier presented in a concise instructional form by ancient teachers. For those who cannot comprehend the geometric demonstration, to them, this algebraic demonstration is to be presented.

It is in his fairly detailed auto-commentary on *Siddhāntaśiromaṇi,* known as the *Mitākṣarā* or the *Vāsanābhāṣya,* that Bhāskara presents detailed explanations as well as *upapattis* for most of the results and procedures outlined in the text. After the invocatory verse addressed to the Sun, Bhāskara states that the overall purpose of his commentary is to explain, to the uninitiated, the deeper aspects of the various topics discussed in the text so that, by practising the discipline of providing proper explanations, they are able to understand the true nature of the science. He also notes that the explanations cannot be really understood without knowledge of the spherics (*gola*) [SiŚi1981, p. 3]:

> अथ निजकृतशास्त्रे तत्प्रसादात् पदार्थान् शिशुजनघृणयाऽहं व्यञ्जयाम्यत्र गूढान् ।
> विमलितमनसां सद्वासनाभ्यासयोगैः हृदि भवति यथैषां तत्त्वभूतार्थबोधः ॥
>
> वासनावगतिर्गोलानभिज्ञस्य न जायते ।
>
> *atha nijakṛtaśāstre tatprasādāt padārthān*
> *śiśujanaghṛṇayā'haṃ vyañjayāmyatra gūḍhān |*
> *vimalitamanasāṃ sadvāsanābhyāsayogairhṛdi bhavati*
> *yathaiṣāṃ tattvabhūtārthabodhaḥ ||*
>
> *vāsanāvagatirgolānabhijñasya na jāyate |*
>
> Then, with His (Sūrya's) blessings, and out of compassion for the novices, I will hereby explain all that is subtle or hidden in the treatise composed by me, so that, by practising the discipline of providing proper explanations, their minds are cleansed (of doubts and misconceptions) and enabled to comprehend the true nature of things.

One who is not familiar with spherics (*golānabhijña*) will not be able to comprehend these explanations.

The *Mitākṣarā* or the *Vāsanābhāṣya* on *Siddhāntaśiromaṇi* is indeed a seminal work, which also serves as a basic sourcebook for understanding the methodology of the science of astronomy in the Indian tradition. In this article, we shall highlight this aspect of the *Vāsanābhāṣya* by focussing on some of the topics dealt with in the second part, *Golādhyāya* (the section on spherics), of *Siddhāntaśiromaṇi.* The *Golādhyāya* itself is meant to clarify the methods outlined in the first part of the book, *Grahagaṇita* (the section on computation of the longitudes and latitudes of the planets). As we shall see, the *Vāsanābhāṣya* on *Golādhyāya* is full of interesting demonstrations and discussions which serve to illustrate the methodological approach of Indian astronomers in the twelfth century.

2 Bhāskara on *upapatti*

2.1 The raison d'être of upapatti in the Indian tradition

We have already noted how, in the *Bījagaṇita-vāsanā,* Bhāskara has referred to the tradition of *upapatti* and also presented a couple of examples of such *upapattis.* In the beginning of the *Golādhyāya* of *Siddhāntaśiromaṇi,* Bhāskara presents what may be regarded as the *raison d'être* of *upapatti* in the Indian tradition of mathematics and astronomy [SiŚi1981, p. 326]:

मध्याद्यं द्युसदां यदत्र गणितं तस्योपपत्तिं विना
प्रौढिं प्रौढसभासु नैति गणको निःसंशयो न स्वयम् ।
गोले सा विमला करामलकवत् प्रत्यक्षतो दृश्यते
तस्मादस्म्युपपत्तिबोधविधये गोलप्रबन्धोद्यतः ॥

madhyādyaṃ dyusadāṃ yadatra gaṇitaṃ tasyopapattiṃ vinā
prauḍhiṃ prauḍhasabhāsu naiti gaṇako niḥsaṃśayo na svayam |
gole sā vimalā karāmalakavat pratyakṣato dṛśyate
tasmādasmyupapattibodhavidhaye golaprabandhodyataḥ ||

Without the knowledge of *upapattis,* by merely mastering the calculations (*gaṇita*) described here, from the *madhyamādhikāra* (the first chapter of *Siddhāntaśiromaṇi*) onwards of the (motion of the) heavenly bodies, a mathematician will not be respected in the scholarly assemblies; without the *upapattis* he himself will not be free of doubt (*niḥsaṃśaya*). The *upapatti* is clearly perceivable in the (armillary) sphere like a berry in the hand; hence, I am embarking on the section on spherics in order to offer a systematic instruction of the *upapattis.*

We find almost the same characterization of *upapatti* being repeated by Gaṇeśa Daivajña in the introduction to his commentary *Buddhivilāsinī* (c. 1545) on *Līlāvatī* [Līlā1937, p. 1]:

> व्यक्ते वाऽव्यक्तसंज्ञे यदुदितमखिलं नोपपत्तिं विना तत्
> निर्भ्रान्तो वा ऋते तां सुगणकसदसि प्रौढतां नैति चायम् ।
> प्रत्यक्षं दृश्यते सा करतलकलितादर्शवत् सुप्रसन्ना
> तस्मादग्र्योपपत्तिं निगदितुमखिलम् उत्सहे बुद्धिवृद्ध्यै ॥
>
> *vyakte vā'vyaktasaṃjñe yaduditamakhilaṃ nopapattiṃ vinā tat*
> *nirbhrānto vā ṛte tāṃ sugaṇakasadasi prauḍhatāṃ naiti cāyam |*
> *pratyakṣaṃ dṛśyate sā karatalakalitādarśavat suprasannā*
> *tasmādagryopapattiṃ nigaditumakhilam utsahe buddhivṛddhyai ||*
>
> Whatever has been stated in the section referred to as computations with known quantities (*vyakta-gaṇita*) or [in the section on] computations with unknown quantities (*avyakta-gaṇita*), it will not be free of misconceptions (*nirbhrānta*) unless it is accompanied by demonstration (*upapatti*); without that (*upapatti*), it will not find acceptance in the assembly of scholarly mathematicians. That [*upapatti*] is directly perceivable as a mirror held in hand. Hence, I am eager to present the foremost *upapatti* in entirety for the sake of enhancement of the intellect.

Thus, according to Bhāskara and Gaṇeśa, the purpose of *upapatti* in the Indian tradition is mainly:

(i) To remove confusion and doubts regarding the validity and interpretation of mathematical results and procedures.

(ii) To obtain assent in the community of mathematicians.

This is very different from the ideal of "proof" in the Greco-European tradition in mathematics and mathematical astronomy, which is to irrefutably establish the absolute truth of a proposition.[2]

The *Vāsanābhāṣya* presents detailed *upapattis* for all the results and processes discussed in the *Siddhāntaśiromaṇi*. Thus, the *Vāsanābhāṣya* also includes a large number of *upapattis* of purely mathematical results, such as the formulae for the surface area and the volume of a sphere ([SiŚi1981, pp. 360–364], see also [Hay1997b, pp. 194–238] and [RM2010, pp. 201–286; esp. 231–235]), or the properties of various spherical triangles which are routinely met with in spherical astronomy.[3] The majority of the *upapattis* found

[2] For a discussion on the nature of proofs in Indian mathematics, see [Sri2005, pp. 209–250].

[3] See for example the *Vāsanābhāṣya* derivation [SiŚi1981, p. 192] of the relation between the *chāyākarṇa* (the hypotenuse of the shadow, related to the zenith distance) and the *digjyā, agrā* and *palabhā,* which are dependent on the declination, azimuth and the latitude of the place, as discussed in the chapter by Prof. M. S. Sriram published in this volume.

in the *Vāsanābhāṣya,* however, pertain to results and procedures that are associated with the analysis of planetary motion. Some of the interesting *upapattis,* which have been discussed recently, are those regarding the occurrence of the omitted lunar months (*kṣayamāsas*),[4] and the computation of the true velocity (*sphuṭagati*) of a planet.[5]

2.2 Upapatti for the revolution numbers (bhagaṇas) of planets

Right at the beginning of the *Grahagaṇitādhyāya* of *Siddhāntaśiromaṇi,* the *Vāsanābhāṣya* addresses the somewhat subtle issue of providing *upapatti* or justification for the various parameters and the computational procedures which are employed in mathematical astronomy for discussing planetary motion. The *Bhagaṇādhyāya* of *Madhyamādhikāra* starts with the number of revolutions (*bhagaṇas*) in a *kalpa* (1,000 *mahāyugas* or 4.32×10^9 years) completed by different planets (*grahas*), their apsides (*uccas*) and nodes (*pātas*). In his *Vāsanā,* Bhāskara presents a detailed analysis of the nature of *upapatti* in mathematical astronomy [SiŚi1981, p. 30]:

अत्रोपपत्तिः। सा तु तत्तद्भाषाकुशलेन तत्तत्क्षेत्रसंस्थानज्ञेन श्रुतगोलेनैव श्रोतुं शक्यते नान्येन। ग्रहमन्दशीघ्रोच्चपाताः स्वस्वमार्गेषु गच्छन्त एतावन्तः पर्ययान् कल्पे कुर्वन्तीत्यत्रागम एव प्रमाणम्। स चागमो महता कालेन लेखकाध्यापकाध्येतृदोषैर्बहुधा जातस्तदा कतमस्य प्रामाण्यम्।

अथ यद्येवमुच्यते गणितस्कन्ध उपपत्तिमानेवागमः प्रमाणम्। उपपत्त्या ये सिध्यन्ति भगणास्ते ग्राह्याः। तदपि न। यतोऽतिप्राज्ञेन पुरुषेणोपपत्तिर्ज्ञातुमेव शक्यते। न तया तेषां भगणानामियत्ता कर्तुं शक्यते। पुरुषायुषाल्पत्वात्। उपपत्तौ तु ग्रहः प्रत्यहं यन्त्रेण वेध्यः। भगणान्तं यावत्। एवं शनैश्चरस्य तावद्वर्षाणां त्रिंशता भगणः पूर्यते। मन्दोच्चानां तु वर्षशतैरनेकैः। अतो नायमर्थः पुरुषसाध्य इति।

अत एवातिप्राज्ञा गणकाः साम्प्रतोपलब्ध्यनुसारिणं प्रौढगणकस्वीकृतं कमप्यागम-मङ्गीकृत्य ग्रहगणित आत्मनो गणितगोलयोर्निरतिशयं कौशलं दर्शयितुं तथाऽन्यैर्भ्रान्ति-ज्ञानेनान्यथोदितानर्थांश्च निराकर्तुमन्यान् ग्रन्थान् रचयन्ति। ग्रहगणित इतिकर्तव्यतायामस्माभिः कौशलं दर्शनीयं भवत्वागमो योऽपि कोऽप्याशयस्तेषाम्। यथाऽत्र ग्रन्थे ब्रह्मगुप्तस्वी-कृतागमोऽङ्गीकृत इति।

तर्हि तिष्ठतु तावदुपपत्त्या भगणानामियत्तासाधनम्। अथ यद्युपपत्तिरुच्यते तर्हि इतरे-तराश्रयदोषशङ्कया वक्तुमशक्या। तथापि संक्षिप्तामुपपत्तिं वक्ष्यामः। इतरेतराश्रयदोषोऽत्र दोषाभासः। उपपत्तिभेदानां यौगपद्येन वक्तुमशक्यत्वात्।

atropapattiḥ | sā tu tattadbhāṣākuśalena tattatkṣetrasaṃsthānajñena śrutagolenaiva śrotuṃ śakyate nānyena | grahamandaśīghroccapātāḥ svasvamārgeṣu gacchanta etā-

[4] Michio Yano, 'Bhāskara II and *Kṣayamāsa*', published in this volume.

[5] See '*Grahagaṇitādhyāya of Bhāskarācārya*' by M. S. Sriram in this volume.

vantaḥ paryayān kalpe kurvantītyatrāgama eva pramāṇam | sa cāgamo mahatā kālena lekhakādhyāpakādhyetṛdoṣairbahudhā jātastadā katamasya prāmāṇyam | atha yadyevamucyate gaṇitaskandha upapattimānevāgamaḥ pramāṇam | upapattyā ye sidhyanti bhagaṇāste grāhyāḥ | tadapi na | yato'tiprājñena puruṣeṇopapattirjñātumeva śakyate | na tayā teṣāṃ bhagaṇānāmiyattā kartuṃ śakyate | puruṣāyuṣālpatvāt | upapattau tu grahaḥ pratyahaṃ yantreṇa vedhyaḥ | bhagaṇāntaṃ yāvat | evaṃ śanaiśvarasya tāvadvarṣāṇāṃ triṃśatā bhagaṇaḥ pūryate | mandoccānāṃ tu varṣaśatairanekaiḥ | ato nāyamarthaḥ puruṣasādhya iti |

ata evātiprājñā gaṇakāḥ sāmpratopalabdhyanusāriṇaṃ prauḍhagaṇakasvīkṛtaṃ kamapyāgamamaṅgīkṛtya grahagaṇita ātmano gaṇitagolayoratiśayaṃ kauśalaṃ darśayituṃ tathā'nyairbhrāntijñānenānyathoditānarthāṃśca nirākartumanyān granthān racayanti | grahagaṇita itikartavyatāyāmasmābhiḥ kauśalaṃ darśanīyaṃ bhavatvāgamo yo'pi ko'pyāśayasteṣām | yathā'tra granthe brahmaguptasvīkṛtāgamo'ṅgīkṛta iti |

tarhi tiṣṭhatu tāvadupapattyā bhagaṇānāmiyattāsādhanam | atha yadyupapattirucyate tarhi itaretarāśrayadoṣaśaṅkayā vaktumaśakyā | tathāpi saṃkṣiptāmupapattiṃ vakṣyāmaḥ | itaretarāśrayadoṣo'tra doṣābhāsaḥ | upapattibhedānāṃ yaugapadyena vaktumaśakyatvāt |

Here, the *upapatti* (for the revolution numbers of the planets etc.): That can only be heard (received) by one who has a mastery over those languages, and knows the locations of the various figures, and has learnt the spherics; not by anyone else. Here, only a reliable textual tradition (*āgama*) is a source of validation (*pramāṇa*) for the claim that the planets (*grahas*), their slow and fast apsides (*mandoccas, śīghroccas*) and their nodes (*pātas*), while moving in their own orbits, execute a given number of revolutions in a *kalpa.*

That textual tradition has become multi-fold, due to the passage of time and due to the errors of the writers, teachers and the students. Then which textual tradition is to be accepted as a source of validation? Then, if (in answer to the above), the following be stated: In the computational branch (*gaṇitaskandha*) [of astral sciences] only a traditional text which is supported by justification (*upapatti*) is a valid source of knowledge (*upapattimān eva āgamaḥ pramāṇam*); and only those revolutions which are supported by justification have to be adopted. Even that is not the case. Because, even a person with extraordinary intellect can hardly know the justification. With that, it is not possible to exactly determine these revolutions (*bhagaṇas*); because of the limited span of the life of humans. In providing justification (*upapatti*), the planet has to be observed daily through instruments; (that must be done) till the revolution is completed. In this way, in the case of Saturn, the revolution is completed in thirty years; in the case of the slow apsides (*mandoccas*), in several hundreds of years. Hence, [it may be said that] this issue is beyond human capacities.

It is only because of the above (limitations) that the highly proficient astronomers, having adopted a traditional text, whatever it may be so far as it gives results which are currently in agreement with observations and is also accepted by (other) competent astronomers, and go ahead and compose other (newer) treatises on the computation of planetary motions, in order to display their unsurpassed skills in computation and spherics, and to refute the erroneous statements of others which are caused by their confusion. Their desire is that 'we should display our skills in the procedures of computing planetary motions, whatever may be the traditional

> text that has been adopted'. Just as in this treatise, the tradition adopted by Brahmagupta is accepted.
>
> Be that as it may, as regards the determination of the precise number of revolutions (*bhagaṇas*) through *upapatti*. Now, if the *upapatti* is to be stated, it may still not be possible, because, it is probably vitiated by the error of mutual dependency (*itaretarāśrayadoṣa*). Even then, we shall go ahead and present the *upapattis* briefly. Actually, here, the error of mutual dependency is only apparent; since it is not possible to state the different kinds of *upapattis* together at once.

While the results and computations in mathematics can be justified or validated, at least in principle, without taking recourse to any tradition or text (*āgama*), the parameters and the theoretical models in mathematical astronomy are founded on an observational and theoretical tradition which is passed on from generation to generation via reliable texts. Thus, Bhāskara also starts with the statement that only a reliable textual tradition (*āgama*) is a source of validation (*pramāṇa*) for the revolution numbers of planets. However, the texts may become corrupted or incomprehensible over time, and so only a textual tradition which is also supported by justification can be accepted as a valid source (*upapattimān eva āgamaḥ pramāṇam*). Here again, the issue arises that if the *upapatti* or justification is via one's own observation, then it is not feasible, since some of the apsides (*mandoccas*) take several hundred years to complete a revolution. Hence, Bhāskara concludes that, under the circumstances, what a highly proficient astronomer does is to adopt a reputed text, which is also accepted by other experts, and also 'gives results which are currently in agreement with observations' (*sāmpratopalabdhyanusārī*). Then, he might also come up with a new treatise to clarify various misconceptions and add refinements wherever possible. Bhāskara declares that he has, in this way, adopted the tradition accepted by Brahmagupta (as expounded in his *Brāhmasphuṭasiddhānta*).

Even after adopting a reliable and reputed text which gives results in conformity with observations, Bhāskara is clear that it is still important to provide justifications, wherever possible, for the results and procedures used in mathematical astronomy. Hence, he proceeds with presenting such *upapattis* for the various revolution numbers presented in the text for the planets etc. [SiŚi1981, pp. 30–33].[6]

In this context, Bhāskara also takes note of the objection as to whether these *upapattis* could be vitiated by the fallacy of circularity or mutual dependence (*itaretarāśrayadoṣa*). He dispels this objection by stating that this possibility of circularity is only apparent, and that actually there is no such circularity. Nṛsiṃha in his *Vāsanāvārttika* elucidates this point by noting that

[6] The *upapattis* for the revolution numbers of the Sun, Moon and the lunar apogee are discussed in '*Grahagaṇitadhyāya of Bhāskarācārya*' by M. S. Sriram in this volume.

the doubt regarding mutual dependency arises because, in some of the *upapattis,* the mean planet (*madhyagraha*) is sought to be inferred from the true planet (*spaṣṭa*). The fallacy of mutual dependency would have been applicable if it were the case that the computed true planet was used as an input to infer the mean planet. But in these *upapattis,* one is using the true planet, as observed by instruments, as an input to infer the mean planet. Therefore, the fallacy is only apparent and not really applicable to the present case.[7]

2.3 The status of planetary models in Indian astronomy

After explaining the various *upapattis* given by Bhāskara for the revolution numbers of planets, their apsides etc., Nṛsiṃha Daivajña goes on to present a long discourse [SiŚi1981, pp. 41–48] on different models of planetary motion, focussing in particular on three models which he refers to as 'the model of the Yavanas' (*yāvanamatam*), 'the model of (Vṛddha) Āryabhaṭa', and 'our model' (*asmanmatam*), the last one being (his formulation) of the planetary theory as expounded in *Brāhmasphuṭasiddhānta* and *Siddhāntaśiromaṇi.* While Nṛsiṃha gives various arguments to show why the model of the Yavanas and that of Āryabhaṭa should be rejected, he still concludes that all the models are justified (*matatrayamapi yuktiyuktam*) as long as they are consistent with the observed motion [SiŚi1981, p. 48]:

> ननु वस्तुनि विकल्पासंभवात्[8] कथं परस्परविरुद्धस्य गतिकारणप्रतिपादकशास्त्रस्य प्रामाण्यम्। उच्यते। ग्रहगतिप्रतिपादकं शास्त्रं तावत्प्रमाणं गतेः प्रत्यक्षोपलब्धत्वात्। तत्कारणं पुरुषबुद्धिप्रभवत्वाद[तात्विकं भवतु] तथाऽपि न कोऽपि दोषः फलतो दोषाभावात्। तत्त्वज्ञानोपयोगिन्याः जीवेश्वरादिविभागकल्पनाया अपि पुरुषबुद्धिप्रभवत्वेन विकल्पस्य दृष्टत्वात्।
>
> *nanu vastuni vikalpāsaṃbhavāt kathaṃ parasparaviruddhasya gatikāraṇapratipādakaśāstrasya prāmāṇyam | ucyate | grahagatipratipādakaṃ śāstraṃ tāvatpramāṇaṃ gateḥ pratyakṣopalabdhatvāt | tatkāraṇaṃ puruṣabuddhiprabhavatvāda [tātvikaṃ bhavatu] tathā'pi na ko'pi doṣaḥ phalato doṣābhāvāt | tattvajñānopayoginyāḥ jīveśvarādivibhāgakalpanāyā api puruṣabuddhiprabhavatvena vikalpasya dṛṣṭatvāt |*
>
> Since it is impossible for the object (or reality) to have options [to be other than what it is], how is it that mutually conflicting theories of the cause of [planetary] motion can be said to be valid? We shall explain. A theory of planetary motion

[7] '*gaṇitakarmaṇā siddhaspaṣṭādyadi madhyajñānaṃ tadetaretarāśrayadoṣaḥ syāt | atra yantravedhopalabdhaspaṣṭānmadhyajñānenānyonyāśraya ityabhiprāyeṇoktaṃ doṣābhāsa iti*' | [SiŚi1981, p. 36]

[8] The published edition reads '*vastunirvikalpāsaṃbhavāt*', which does not seem to make sense here.

is valid to the extent that it is consistent with the motion as observed. The cause of such [motion], being a product of human intellect, [may be unreal] but still it cannot be faulted since there are no errors in the results; because, we can see that there are alternate versions even in the conceptions of *Jīva, Īśvara,* etc., which are useful in understanding the ultimate true reality, since they are products of human intellect.

Nṛsiṃha concludes with the following statement of Caturveda Pṛthūdakāsvāmin (c. 850), presumably from his famous commentary on *Brāhmasphuṭasiddhānta,* that the situation in astronomy is no different from other disciplines, such as grammar and medicine, that 'unreal' or 'fictitious' (*asatyarūpa*) theoretical frameworks or other aids are employed in order to explain the real or the observed phenomena:

तथा चाह ब्रह्मगुप्तभाष्यकारश्चतुर्वेदाचार्यः – यथा वैय्याकरणाः प्रकृतिप्रत्ययागमलोपविकारैरसत्यरूपैः सत्यं शब्दसाधुत्वं प्रतिपद्यन्ते। यथा च भिषग्वरा उत्पलनालादिभिः शिरावेधादीन् प्रतिपद्यन्ते तथैव सांवत्सराः फलावलम्बमन्दशीघ्रप्रतिमण्डलादिभिर्ग्रहगतितत्त्वं भूमानादितत्त्वञ्च प्रतिपद्यन्त इति मत्वा सन्तोष्टव्यमिति।

tathā cāha brahmaguptabhāṣyakāraścaturvedācāryaḥ – yathā vaiyyākaraṇāḥ prakṛtipratyayāgamalopavikārairasatyarūpaiḥ satyaṃ śabdasādhutvaṃ pratipadyante | yathā ca bhiṣagvarā utpalanālādibhiḥ śirāvedhādīn pratipadyante tathaiva sāmvatsarāḥ phalāvalambamandaśīghrapratimaṇḍalādibhirgrahagatitattvaṃ bhūmānāditattvañca pratipadyanta iti matvā santoṣṭavyamiti |

It has been said by Caturvedācārya (Pṛthūdakasvāmin), the commentator of Brahmagupta: Just as the grammarians employ fictitious entities such as *prakṛti, pratyaya, āgama, lopa* and *vikāra,* to arrive at the validity of the utterances as observed in reality, and just as the proficient doctors employ lotus stalk etc., to practise venesection etc., we have to realise and feel contented that it is only in the same manner that the astronomers employ result-dependent (*phalāvalamba*) [notions such as the] *manda* and *śīghra* eccentrics (*pratimaṇḍala*), etc., to arrive at the true nature of the motion of planets, and the size etc., of the earth.

Similar views have been expressed, in fact much earlier, by the savants of the Āryabhaṭa-school also. While explaining the model of planetary motion as indicated in the *Kālakriyāpāda* (Verse 17) of *Āryabhaṭīya,* Bhāskara I (c. 629) states in his *Āryabhaṭīyabhāṣya* [AB1976, p. 217]:

उच्चनीचमध्यमपरिधिरित्येवमादिस्फुटगतिसाधनोपाय [भूतानाञ्च] उपायानां नैव नियमोक्तिर्वा विद्यते। केवलं तु उपेयसाधका उपायाः। तस्मादियं सर्वा प्रक्रिया असत्या यया ग्रहाणां स्फुटगतिः साध्यते। एवं च परमार्थजिज्ञासुभिः असत्योपायेन सत्यं प्रतिपद्यते। तथा हि भिषजो ह्युत्पलनालादिषु वेधादीन्यभ्यस्यन्ते नापिताः पिठरादिषु मुण्डनादीनि यज्ञशास्त्रविदः शुष्केष्ट्या यज्ञादीनि शाब्दिकाः प्रकृतिप्रत्ययविकारागमवर्णलोपव्यत्ययादिभिः शब्दान् प्रतिजानते। एवमत्रापि मध्यममन्दोच्चशीघ्रोच्चतत्परिधिज्याकाष्ठभुजाकोटिकर्णादिव्यवहारेण सांवत्सरा ग्रहाणां स्फुटगतिं प्रतिजानते। तस्मात् उपायेष्वसत्येषु सत्यप्रतिपादनपरेषु न चोद्यमस्ति।

uccanīcamadhyamaparidhirityevamādisphuṭagatisādhanopāya [bhūtānāñca] upāyānāṃ naiva niyamoktirvā vidyate | kevalaṃ tu upeyasādhakā upāyāḥ |

tasmādiyaṃ sarvā prakriyā asatyā yayā grahāṇāṃ sphuṭagatiḥ sādhyate | evaṃ ca paramārthajijñāsubhiḥ asatyopāyena satyaṃ pratipadyate | tathā hi bhiṣajo hyutpalanālādiṣu vedhādīnyabhyasyante nāpitāḥ piṭharādiṣu muṇḍanādīni yajñaśāstravidaḥ śuṣkeṣṭyā yajñādīni śābdikāḥ prakṛtipratyayavikārāgamavarṇalopavyatyayādibhiḥ śabdān pratijānate | evamatrāpi madhyamamandoccaśīghroccatatparidhijyākāṣṭhabhujākoṭikarṇādivyavahāreṇa sāṃvatsarā grahāṇāṃ sphuṭagatiṃ pratijānate | tasmādupāyeṣvasatyeṣu satyapratipādanapareṣu na codyamasti |

There are no constraints or limitations imposed on the means such as the *ucca, nīca, madhyama, paridhi* and so on which are essentially aids to the calculation of the true motion of the planets. They are only the means (*upāya*) for arriving at the desired objectives (results). Hence this entire procedure is fictitious, by means of which the true motion of planets is arrived at. Also, in the same way, the seekers of ultimate reality arrive at the truth by taking recourse to false means. Similarly, the doctors practise surgery etc., by employing lotus stalks; the barbers (practise) shaving with leafy surfaces etc.; the experts in the science of sacrifice (practise) sacrifices etc., with dry bricks; the linguists employ notions such as *prakṛti, pratyaya, vikāra, āgama, varṇa, lopa, vyatyaya,* etc., to comprehend (well-formed) utterances. In the same way, here also the astronomers comprehend the true motion of planets by employing notions such as *madhyama, mandocca, śīghrocca, śīghraparidhi, jyā, kāṣṭha, bhujā, koṭi, karṇa,* etc. Hence, there is indeed nothing unusual that fictitious means are employed to arrive at the true state of affairs (in all these disciplines).

The above statements clearly elucidate the pragmatic and open-ended approach to scientific theorisation that is characteristic of the scientific tradition in India, which is strikingly different from the quest for absolutely true universal laws which has dominated the Greco-European scientific tradition.[9] In this context, the Indian texts often refer to the famous dictum of Bhaṛtṛhari, which is also cited by Nīlakaṇṭha Somayājī in his *Āryabhaṭīyabhāṣya,* [AB1931, p. 31] that the theoretical framework and procedures taught in the *śāstras* are only means (*upāya*) for accomplishing the given objectives, and there should be no other constraint imposed upon them [VāPa1980, p. 78]:

उपायानाञ्च नियमो नावश्यमवतिष्ठते ।

upāyānāñca niyamo nāvaśyamavatiṣṭhate |

There is no limitation that has to be imposed on the means (*upāyas*) (the procedures taught in *Śāstras* for accomplishing the objectives laid down).

While explaining the above dictum, which was made in the context of the science of grammar (*vyākaraṇaśāstra*), the commentator Puṇyarāja observes that [VāPa1980, p. 81]:

कश्चिदाचार्यः पाणिनिविरचितेन लक्षणशास्त्रेण शब्दानधिगच्छति कश्चिदन्येनेति न नियमः ।

[9] For further discussion on the pragmatic approach to scientific theorization in Indian tradition see [Baj1988] and [Sri2015].

kaścidācāryaḥ pāṇiniviracitena lakṣaṇaśāstreṇa śabdānadhigacchati kaścidanyeneti na niyama: |

One preceptor (*Ācārya*) understands utterances by means of the grammatical framework of Pāṇini and another by means of another framework and thus there is no rule [that only a particular grammar is to be followed].

Thus, the Indian grammarians are not willing to commit to the uniqueness and universality of even Pāṇini's *Aṣṭādhyāyī,* whose efficacy as a grammar has been universally acclaimed.

3 The planetary model of Bhāskara[10]

3.1 Computation of the true planet

The method of computing the true longitudes of planets is discussed in the *Spaṣṭādhikāra* of *Grahagaṇitādhyāya* [SiŚi1981, pp. 99–141].[11] The actual process of computation of the true planet is discussed in verses 34–35. As in all the other texts, only the *manda*-correction is prescribed in the case of the Sun and the Moon; and a combination of the *manda* and *śīghra* corrections is prescribed for the five planets. While combining the *manda* and *śīghra* corrections, Bhāskara prescribes a two-step process (with variations in the case of Mars), which is to be iterated till final result remains unchanged (*aviśeṣa*) [SiŚi1981, p. 118]:

[10] For a detailed discussion of planetary models in Indian astronomy see [TaSa2011, pp. 487–535].

[11] In the first verse, Bhāskara emphasises that it is the true longitudes which are relevant in all practical applications, and hence the procedure for the computation of true longitudes (*sphuṭakriyā*) should be so as to achieve consonance of the computations with observations (*dṛgaṇitaikyakṛt*). Verses 2–17 deal with the computation of Rsines (*jyānayana*) and the computation of the arc (*dhanus*) from the Rsine. The verses 18–21 deal with the *manda* and *śīghra* anomalies in different quadrants. Verse 22 lists the *manda*-epicycles for the Sun, Moon and the planets and the verses 23–25 list the *śīghra*-epicycles of the planets and some corrections to be made for the *manda* and *śīghra* epicycles of Venus and Mars depending on their anomalies.

In verses 26–29, 32–33, and the *Vāsanābhāṣya* thereon, Bhāskara explains the eccentric (*pratimaṇḍala*) and the epicycle (*nīcoccavṛtta*) models for arriving at the corrections (*phala*), the different expressions for the hypotenuse (*karṇa*) in both the models, as also the final expressions for the *manda* and *śīghra*-corrections. As prescribed in the *Brāhmasphuṭasiddhānta* as well as in the texts of the Āryabhaṭa school, the expressions for both the *manda*-correction and the *śīghra*-correction are identical (in terms of the corresponding anomalies and epicycles), except that in the case of the *manda*-correction, instead of the hypotenuse (*karṇa*) only the radius of the concentric appears in the denominator.

स्यात् संस्कृतो मन्दफलेन मध्यो मन्दस्फुटोऽस्माच्चलकेन्द्रपूर्वम् ॥
विधाय शैघ्र्येण फलेन चैवं खेटः स्फुटः स्यादसकृत् फलाभ्याम् ।
दलीकृताभ्यां प्रथमं फलाभ्यां ततोऽखिलाभ्यामसकृत् कुजस्तु ॥
स्फुटौ रवीन्दू मृदुनैव वेद्यौ शीघ्राख्यतुङ्गस्य तयोरभावात् ।

syāt saṃskṛto mandaphalena madhyo mandapuṭo'smāccalakendrapūrvam ||
vidhāya śaighryeṇa phalena caivaṃ kheṭaḥ sphuṭaḥ syādasakṛt phalābhyām |
dalīkṛtābhyāṃ prathamaṃ phalābhyāṃ tato'khilābhyāmasakṛt kujastu ||
sphuṭau ravīndū mṛdunaiva vedyau śīghrākhyatuṅgasya tayorabhāvāt |

The mean planet corrected by *manda*-correction will be the *manda-sphuṭa*. From this (*manda-sphuṭa*), first the *śīghra*-anomaly is found and it (*manda-sphuṭa*) is corrected by the *śīghra*-correction; the true planet is obtained by successive iteration of both these corrections. In the case of Mars, initially half of these two corrections are applied and then the whole of them are applied iteratively. Since there is no *śīghrocca* for the Sun and the Moon, it should be understood that in their case the true planet will result by the application of the *manda*-correction alone.

In his *vāsanā,* Bhāskara explains the steps in the above process. He also refers to the explanation given in the *Golādhyāya* that the *manda*-correction is first carried out to know the position of the centre of the *śīghra*-epicycle. For the special procedure that is employed in the case of Mars, Bhāskara states that the only justification is that it is in accordance with what is observed [SiŚi1981, p. 118]:

आदौ ग्रहस्य मन्दफलमानीय तेन संस्कृतोऽसौ मन्दस्फुटः स्यात्। तं शीघ्रोच्चाद्विशोध्य शीघ्रकेन्द्रं कृत्वा ततः शीघ्रफलं तेन संस्कृतो मन्दस्फुटो ग्रहः स्फुटः स्यात्। तस्मात् स्फुटान्मन्दोच्चं विशोध्य मन्दफलमानीय तेन गणितागतो मध्यः संस्कृतो मन्दस्फुटः स्यात्। तेन पुनश्चलकेन्द्रं ततश्चलफलं तेन मन्दस्फुटः संस्कृतः स्फुटः स्यात्। एवमसकृद्यावदविशेषः। अस्योपपत्तिर्गोले।

शीघ्रनीचोच्चवृत्तस्य मध्यस्थितिं ज्ञातुमादौ कृतं कर्म मान्दं ततः ।
खेटबोधाय शैघ्र्यं मिथःसंश्रिते मान्दशैघ्र्ये हि तेनासकृत् साधिते ॥

तथा मन्दकर्मणि कर्णो न कृतस्तत्कारणमपि गोले कथितम्। यत् तु `दलीकृताभ्यां प्रथमं फलाभ्याम्' इत्यादि कुजस्य विशेषः तत्रोपलब्धिरेव वासना।

ādau grahasya mandaphalamānīya tena saṃskṛto'sau mandasphuṭaḥ syāt | taṃ śīghroccādviśodhya śīghrakendraṃ kṛtvā tataḥ śīghraphalaṃ tena saṃskṛto mandasphuṭo grahaḥ sphuṭaḥ syāt | tasmāt sphuṭānmandoccaṃ viśodhya mandaphalamānīya tena gaṇitāgato madhyaḥ saṃskṛto mandasphuṭaḥ syāt | tena punaścalakendraṃ tataścalaphalaṃ tena mandasphuṭaḥ saṃskṛtaḥ sphuṭaḥ syāt | evamasakṛdyāvadaviśeṣaḥ | asyopapattirgole |

śīghranīcoccavṛttasya madhyasthitiṃ
jñātumādau kṛtaṃ karma māndaṃ tataḥ |
kheṭabodhāya śaighryaṃ mithaḥsaṃśrite
māndaśaighrye hi tenāsakṛt sādhite ||

tathā mandakarmaṇi karṇo na kṛtastatkāraṇamapi gole kathitam | yat tu 'dalīkṛtābhyāṃ prathamaṃ phalābhyām' ityādi kujasya viśeṣastatropalabdhireva vāsanā |

First obtain the *manda*-correction (*manda-phala*) of the planet and this (mean planet) corrected by that (correction) will be the *manda-sphuṭa*. Deduct that (*manda-sphuṭa*) from the *śīghrocca* to get the *śīghra*-anomaly (*śīghra-kendra*) and, using that, obtain the *śīghra*-correction (*śīghra-phala*). The *manda-sphuṭa* corrected by that (*śīghra-phala*) will be the (first approximation to the) true planet (*sphuṭa*). From that *sphuṭa* subtract the *mandocca* and obtain the *manda*-correction; the mean planet corrected by that will be the corrected *manda-sphuṭa*. From that the *śīghra*-anomaly is found, and from that the *śīghra*-correction and the (corrected) *manda-sphuṭa* when corrected by that (*śīghra*-correction) will be the corrected true planet (*saṃskṛta-sphuṭa*). This (entire process) is to be iterated till there is no difference (between the successive results). The *upapatti* for this (is presented) in the *Golādhyāya*.

> In order to know the location of the centre of the *śīghra*-epicycle, initially the *manda*-process is carried out. Then, to find the true planet the *śīghra*-process (is carried out). Since the *manda* and *śīghra*-processes are mixed together (*mithaḥsaṃśrite*), they are to be carried out iteratively.

In the same way, the reason why the hypotenuse (*karṇa*) is not evaluated in the *manda*-process will be stated in the *Golādhyāya*. For the special procedure employed in the case of Mars, as stated in the verse 'initially half of these two corrections…' ('*dalīkṛtābhyāṃ prathamaṃ phalābhyam…*'), the only justification is that is what is in conformity with observations (*upalabdhireva tatra vāsanā*).

3.2 Bhāskara follows Brahmagupta in the use of iterated manda-hypotenuse

In the case of the *manda*-correction, Bhāskara follows the variable epicycle model employed by both Āryabhaṭa (as explained by his commentator Bhāskara I) and Brahmagupta, according to which: The tabulated epicycle radii are only the mean values and the true epicycle radius increases and decreases in proportion to the hypotenuse (*karṇa*), and the true epicycle radius and the hypotenuse are to be obtained by a process of iteration (*asakṛt-karma, aviśeṣa-karma*). We shall first briefly outline the *manda*-correction process and the notion of iterated *manda*-hypotenuse (*aviśiṣṭa-manda-karṇa*) as discussed by Bhāskara I and Brahmagupta.[12]

In Figure (1), O is the centre of the earth, P_0 the mean planet and U the *mandocca*. $OP_0 = R$, is the radius of the concentric. P_1 is on the epicycle centred at P_0, with radius equal to the mean or tabulated epicycle radius r_0, such that P_0P_1 is parallel to OU. P_1 is also on the eccentric circle with

[12] For more details see [MaBk1960, pp. 111–119]. Also, [TaSa2011, pp. 494–497].

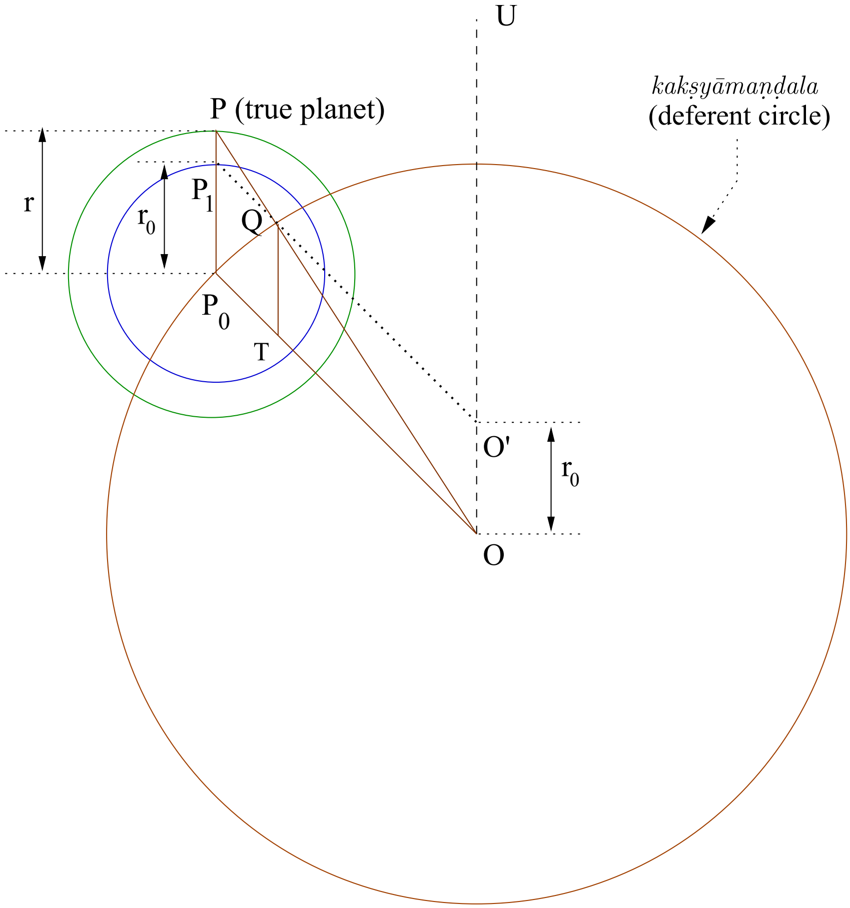

Figure 1: *Manda*-correction and the iterated-*manda*-hypotenuse.

O' as the centre, where O' is along OU, such that $OO' = r_0$. Here, P_1 is not the *manda*-corrected planet or the *manda-sphuṭa*. $OP_1 = K_0$, is only the initial hypotenuse or the *sakṛt-karṇa*. The *manda-sphuṭa* is at P (along P_0P_1), such that the true epicycle radius $P_0P = r$ and the true *manda*-hypotenuse $OP = K$, are related by

$$r = \left(\frac{r_0}{R}\right) K. \tag{1}$$

Let θ_0 be the longitude of the mean planet P_0, θ_u the longitude of the *mandocca* U, and θ_{ms} the longitude of the *manda-sphuṭa* P. From Figure (1), and

the above condition (1), we can easily see that the *manda*-correction for the longitude will be

$$R\sin(\theta_{ms} - \theta_0) = \left(\frac{r}{K}\right) R\sin(\theta_0 - \theta_u) = \left(\frac{r_0}{R}\right) R\sin(\theta_0 - \theta_u). \qquad (2)$$

Thus, the *manda*-correction (2) involves only the ratio of the mean epicycle radius r_0 and the concentric. It does not involve the initial hypotenuse K_0 or even the true hypotenuse K.

In order to determine both r and K, the following iterative process is employed. To start with, the initial hypotenuse (*sakṛt-karṇa*), K_0, is computed in the usual way in terms of the anomaly, using the mean epicycle radius r_0:

$$K_0 = OP_1 = \left[\{R\sin(\theta_0 - \theta_u)\}^2 + \{R\cos(\theta_0 - \theta_u) + r_0\}^2\right]^{\frac{1}{2}}. \qquad (3)$$

Then the next approximation to the epicycle radius, r_1, is found using

$$r_1 = \left(\frac{r_0}{R}\right) K_0. \qquad (4)$$

From r_1 the corresponding hypotenuse K_1 is computed using

$$K_1 = \left[\{R\sin(\theta_0 - \theta_u)\}^2 + \{R\cos(\theta_0 - \theta_u) + r_1\}^2\right]^{\frac{1}{2}}. \qquad (5)$$

And, from K_2, the next approximation r_2 is computed using

$$r_2 = \left(\frac{r_0}{R}\right) K_1. \qquad (6)$$

And so on, till there is no appreciable difference between successive results (*aviśeṣa*), which means that, for some m

$$r_{m+1} = \left(\frac{r_0}{R}\right) K_m \approx r_m. \qquad (7)$$

Then, it can be seen right away that the iterated radius r_m and the associated hypotenuse K_m, are such that

$$r_m \approx \left(\frac{r_0}{R}\right) K_m. \qquad (8)$$

In other words, they very nearly satisfy the relation (1) that characterises the true epicycle r and the corresponding true hypotenuse K.

Bhāskara has discussed the rationale of the process of *manda*-correction later, in verses 36–37 of the *Chedyakādhyāya* of *Golādhyāya,* and his commentary thereon. In response to the doubt as to why the hypotenuse is not used in *manda*-correction, Bhāskara first refers to the view that it is not used because

the corresponding difference (between K_0 and R) is rather small. He then goes on to discuss the view of Brahmagupta (closely following the verse 29, of Chapter XXI of the *Brāhmasphuṭasiddhānta,* [BSS2003, p. 98][13] *trijyābhaktaḥ karṇaḥ...*), that even when we use the exact formula for the *manda*-correction in terms of the hypotenuse, since the true values of the epicycle radius and the hypotenuse are proportional (as in eq.(1)), the result would be the same (as in eq.(2)), where the hypotenuse gets replaced by the radius of the concentric. Bhāskara also notes that it would not be proper to doubt as to why such a procedure is adopted in the case of *śīghra* correction as the rationale for the corrections is indeed peculiar [SiŚi1981, p. 392]:

> इदानीं मन्दकर्मणि कर्णः किं न कृत इत्याशङ्क्योत्तरमाह –
>
> स्वल्पान्तरत्वान्मृदुकर्मणीह कर्णः कृतो नेति वदन्ति केचित् ।
> त्रिज्योद्धृतः कर्णगुणः कृतेऽपि कर्णस्फुटः[14] स्यात् परिधिर्यतोऽत्र ॥
> तेनाद्यतुल्यं फलमेति तस्मात् कर्णः कृतो नेति न केचिदूचुः ।
> नाशङ्कनीयं न चले किमित्थं यतो विचित्रा फलवासनात्र ॥
>
> *idānīṃ mandakarmaṇi karṇaḥ kiṃ na kṛta ityāśaṅkyottaramāha —*
>
> *svalpāntaratvānmṛdukarmaṇīha karṇaḥ kṛto neti vadanti kecit |*
> *trijyoddhṛtaḥ karṇaguṇaḥ kṛte'pi karṇasphuṭaḥ syāt paridhiryato'tra ||*
> *tenādyatulyaṃ phalameti tasmāt karṇaḥ kṛto neti na kecidūcuḥ |*
> *nāśaṅkanīyaṃ na cale kimitthaṃ yato vicitrā phalavāsanātra ||*
>
> Now he raises the question as to why in the *manda*-process the hypotenuse (*karṇa*) is not evaluated and answers it:
>
> Some say that here in the *manda*-process the hypotenuse is not evaluated because [employing it or not] makes little difference. Some [Brahmagupta and his followers] say that, since here the circumference of (the *manda*-epicycle) is made true through the hypotenuse by (an iterative process of) multiplication by the hypotenuse and division by the radius (of the concentric), the final correction (involving the true hypotenuse) will be the same as the first result (involving only the Radius), and therefore the hypotenuse is not evaluated (in the *manda*-process). It should not then be doubted as to why the same (iterative process) is not followed in the *śīghra*-process, because the rationale of these corrections is indeed peculiar.

In his *vāsanā,* Bhāskara reiterates that Brahmagupta has given the correct explanation of the *manda*-correction process. He also presents a detailed explanation as to why the *manda*-hypotenuse does not appear in the final form of *manda*-correction. We have outlined the same argument, above, in mathematical terms via equations (1)–(8). Bhāskara also notes that it was Pṛthūdaka who raised the objection as to why a similar iterative procedure was

[13] The verse 29 of Chapter XXI, *Golādhyāya,* is: *trijyābhaktaḥ karṇaḥ paridhiguṇo bāhukoṭi guṇakāraḥ. asakṛn mānde tatphalam ādyasamaṃ nātra karṇo'smāt.* The first *pāda* of this verse is actually cited in the *Vāsanābhāṣya* of Bhāskara (see below).

[14] The published versions ([SiŚi1981, p. 392], [SiŚi1943, p. 189] read *karṇe sputaḥ,* which does not seem to make sense here.

not adopted in the śīghra-process, and even went to the extent of accusing Brahmagupta that he was trying to mislead others by advocating the iterative process for finding the true *manda*-epicycle and the hypotenuse. Bhāskara is totally dismissive of Pṛthūdaka's accusation and concludes that it is Brahmagupta's explanation which is indeed appropriate and also appealing.

इह कर्णेन यत्फलमानीयते तदेव समीचीनम्। यन्मन्दकर्मणि कर्णो न कृतः तत् स्वल्पान्तरत्वात्। मन्दफलानि हि स्वल्पानि भवन्ति। तदन्तरं चातिस्वल्पमिति केषाञ्चित् पक्षः।

ब्रह्मगुप्तोऽत्र कारणमाह 'त्रिज्याभक्तः परिधिः कर्णगुण [BSS2003, p. 98] इत्यादि। मन्दकर्मणि मन्दकर्णतुल्येन व्यासार्धेन यद्वृत्तमुत्पद्यते तत् कक्षामण्डलम्। तेन ग्रहो गच्छति। यो मन्दपरिधिः पाठपठितः स त्रिज्यापरिणतः। अतोऽसौ कर्णव्यासार्धे परिणम्यते।[15] ततोऽनुपातः। यदि त्रिज्यावृत्तेऽयं परिधिस्तदा कर्णवृत्ते क इति। अत्र परिधेः कर्णो गुणः त्रिज्या हरः। एवं स्फुटपरिधिः तेन दोर्ज्या गुण्या भांशैः ३६० भाज्या। ततस्त्रिज्यया गुण्या कर्णेन भाज्या। एवं सति त्रिज्यातुल्ययोः कर्णतुल्ययोश्च गुणहरयोस्तुल्यत्वान्नाशे कृते पूर्वफलतुल्यमेव फलमागच्छतीति ब्रह्मगुप्तमतम्। अथ यद्येवं परिधेः कर्णेन स्फुटत्वं तर्हि किं शीघ्रकर्मणि न कृतमित्याशङ्क्य चतुर्वेद आह ब्रह्मगुप्तेनान्येषां प्रतारणपरमिदमुक्तमिति। तदसत्। चले कर्मणीत्थं किं न कृतमिति नाशङ्कनीयं यतः फलवासना विचित्रा। शुक्रस्यान्यथा परिधेः स्फुटत्वं भौमस्यान्यथा तथा किं न बुधादीनामिति नाशङ्क्यम्। अतो ब्रह्मोक्तिरत्र सुन्दरी।

iha karṇena yatphalamānīyate tadeva samīcīnam | yanmandakarmaṇi karṇo na kṛtastat svalpāntaratvāt | mandaphalāni hi svalpāni bhavanti | tadantaraṃ cātisvalpamiti keṣāṃcit pakṣaḥ |

brahmagupto'tra kāraṇamāha 'trijyābhaktaḥ paridhiḥ karṇaguṇa' ityādi | mandakarmaṇi mandakarṇatulyena vyāsārdhena yadvṛttamutpadyate tat kakṣāmaṇḍalam |tena graho gacchati | yo mandaparidhiḥ pāṭhapaṭhitaḥ sa trijyāpariṇataḥ | ato'sau karṇavyāsārdhe pariṇamyate | tato'nupātaḥ | yadi trijyāvṛtte'yaṃ paridhistadā karṇavṛtte ka iti atra paridheḥ karṇo guṇastrijyā haraḥ | evaṃ sphuṭaparidhistena dorjyā guṇyā bhāṃśaiḥ 360 bhājyā | tatastrijyayā guṇyā karṇena bhājyā | evaṃ sati trijyātulyayoḥ karṇatulyayośca guṇaharayostulyatvānnāśe kṛte pūrvaphalatulyameva phalamāgacchatīti brahmaguptamatam | atha yadyevaṃ paridheḥ karṇena sphuṭatvaṃ tarhi kiṃ śīghrakarmaṇi na kṛtamityāśaṅkya caturveda āha brahmaguptenānyeṣāṃ pratāraṇaparamidamuktamiti | tadasat | cale karmaṇītthaṃ kiṃ na kṛtamiti nāśaṅkanīyaṃ yataḥ phalavāsanā vicitrā | śukrasyānyathā paridheḥ sphuṭatvaṃ bhaumasyānyathā tathā kiṃ na budhādīnāmiti nāśaṅkyam | ato brahmoktiratra sundarī |

Some say: 'Here, it is the correction, which is obtained using the hypotenuse, that is the appropriate one. The reason, why the hypotenuse is not employed in the *manda*-correction, is only because it makes little difference (either way). The *manda* corrections are indeed small. So the differences will be very minute'.

In this context, Brahmagupta has stated the (correct) reason [in his verse] 'The circumference (of the epicycle) multiplied by the hypotenuse and divided by the radius' etc. (*trijyābhaktaḥ paridhiḥ karṇaguṇaḥ*). In the *manda* process, the circle that is produced with radius equal to *manda*-hypotenuse, that is the orbit circle. The planet moves on that. The stated (tabulated) circumference of the *manda*-

[15] Here, the published versions ([SiŚi1981, p. 392], [SiŚi1943, p. 189] have the erroneous reading: *pariṇāmyate*.

epicycle (*manda-paridhi*) is in the measure of the *trijyā* (radius of the concentric). Hence it (the *manda-paridhi*) is transformed to be in proportion with the radius of the hypotenuse circle (*karṇavyāsārdha*). Thus, the rule of proportion: If so much is the *manda-paridhi* in the concentric circle then how much will it be in the circle with the hypotenuse as the radius (*karṇavṛtta*). Here, for the (tabulated) *manda-paridhi* the multiplier is the hypotenuse and the divisor is the radius of the concentric. This gives the true *manda-paridhi* which is to be multiplied by the Rsine of the anomaly (*dorjyā*) and divided by 360 degrees. Then it is to be multiplied by the radius of the concentric and divided by the hypotenuse. Since the multipliers and divisors, being equal to the radius of the concentric and the hypotenuse, mutually cancel each other, one arrives at the original expression for the correction. Such is the view of Brahmagupta.

Here, raising the objection that if the true circumference of the epicycle is to be determined by the hypotenuse why the same is not done in the case of *śīghra*-process, Caturveda (Pṛthūdakācārya) said that the above (explanation) has been stated by Brahmagupta to mislead others. That is not true. We should not object as to why the same procedure is not followed in the *śīghra*-process since the rationale of these corrections is indeed peculiar. [Similarly] we should not object as to why the epicycles are corrected one way in the case of Mars, in another way in the case of Venus, and not corrected at all in the case of Mercury, etc. Hence, in this context, what is said by Brahmagupta is indeed appealing (*sundarī*).

3.3 Bhāskara's approximation for the aviśiṣṭa-manda-karṇa

In verse 4 of the chapter on lunar eclipses (*Candragrahaṇādhikāra*) of *Gaṇitādhyāya,* Bhāskara notes that the true distances of the Sun and Moon are different from their mean distances. Their true distances in minutes (*kalākarṇa,* which have to be later converted to *yojanas*) are in fact variable and given by the iterated-*manda*-hypotenuse (*aviśiṣṭa-manda-karṇa*). Bhāskara also gives a single step approximation to this iterated-*manda*-hypotenuse, and claims that it can be directly verified by actual computation [SiŚi1981, p. 231]:

मन्दश्रुतिर्द्राक्श्रुतिवत् प्रसाध्या तया त्रिभज्या द्विगुणा विहीना ।
त्रिज्याकृतिः शेषहृता स्फुटा स्यात् लिप्ताश्रुतिस्तिग्मरुचेर्विधोश्च ॥

इह स्पष्टीकरणे ये मन्दनीचोच्चवृत्तपरिधिभागाः पठिताः ते त्रिज्यातुल्ये कक्ष्याव्यासार्धे। यदा ग्रहस्य कर्ण उत्पन्नस्तदा कर्णो व्यासार्धं ग्रहकक्षायाः। अथ त्रैराशिकेन तत्परिणतास्ते कार्याः। यदि त्रिज्याव्यासार्धे एते मन्दपरिधिभागास्तदा कर्णव्यासार्धे क इति। एवं परिधेः स्फुटत्वं विधाय असकृत् कर्णः कार्यः। स कलाकर्णः स्फुटो भवति।

एतदसकृत्कर्मोपसंहृत्य सकृत्कर्मणा कर्णस्य स्फुटत्वं कृतम्। प्रथमं यः कर्णः आगतः तमेव त्रिज्यारूपं प्रकल्प्य स्फुटः कर्णोऽत्र साध्यते। यदा किल कर्णस्त्रिज्यातो नूनो भवति यावता न्यूनः तत् त्रिज्यया संशोध्य यद्यधिको वर्तते यावताधिकः तत् त्रिज्यया विशोध्य शेषेणानुपातः।

यद्यनेन त्रिज्या लभ्यते तदा त्रिज्यया किमिति। अनेनानुपातेन स्फुटः कर्णः सकृद्भवति। अत्र धूलीकर्मणा प्रत्यक्षप्रतीतिः।

mandaśrutirdrākśrutivat prasādhyā tayā tribhajyā dviguṇā vihīnā |
trijyākṛtiḥ śeṣahṛtā sphuṭā syālliptāśrutistigmarucervidhośca ||

iha spaṣṭīkaraṇe ye mandanīcoccavṛttaparidhibhāgāḥ paṭhitāste trijyātulye kakṣyāvyāsārdhe | yadā grahasya karṇa utpannastadā karṇo vyāsārdhaṃ grahakakṣāyāḥ | atha trairāśikena tatpariṇatāste kāryāḥ | yadi trijyāvyāsārdhe ete mandaparidhibhāgāstadā karṇavyāsārdhe ka iti | evaṃ paridheḥ sphuṭatvaṃ vidhāyāsakṛt karṇaḥ kāryaḥ | sa kalākarṇaḥ sphuṭo bhavati |

etadasakṛtkarmopasaṃhṛtya sakṛtkarmaṇā karṇasya sphuṭatvaṃ kṛtam | prathamaṃ yaḥ karṇaḥ āgatastameva trijyārūpaṃ prakalpya sphuṭaḥ karṇo'tra sādhyate | yadā kila karṇastrijyāto nūno bhavati yāvatā nyūnastat trijyayā saṃśodhya yadyadhiko vartate yāvatādhikastat trijyayā viśodhya śeṣeṇānupātaḥ | yadyanena trijyā labhyate tadā trijyayā kimiti | anenānupātena sphuṭaḥ karṇaḥ sakṛdbhavati | atra dhūlīkarmaṇā pratyakṣapratītiḥ |

The *manda*-hypotenuse should be obtained in the same way as the hypotenuse of the *śīghra*-process (*drākśrutivat*) and that is to be subtracted from twice the radius. The square of the radius, divided by the remainder (of the above), will give the true hypotenuse in minutes in the case of the Sun and the Moon.

Here, in the computation of true planets, the circumferences of the *manda*-epicycles (*manda-paridhi*) which are tabulated in degrees, they are in (relation to) the orbit circle (concentric) whose radius is equal to Rsine of three signs (*trijyā*). When the planet has a (different) hypotenuse, then the hypotenuse is the radius of the orbit of the planet. Hence, the corresponding quantities have to be evaluated by the rule of three. If the given degrees of *manda-paridhi* are associated with the radius of the concentric, then what would it they be when the radius is given by the hypotenuse. In this way, having ensured the correctness of the epicycle radius, the iterated hypotenuse is to be computed. That hypotenuse in minutes (*kalākarṇa*) will be the correct one.

Having concluded (the discussion of) the iterative process, the true hypotenuse is now worked out by a single-step process (*sakṛt-karma*). Taking the hypotenuse obtained in the first step to be the radius, the true hypotenuse is now arrived at. If the hypotenuse (obtained in the first step) happens to be smaller than the radius, the amount, by which it is less, is added to the radius; if more, then the amount by which it is more is subtracted from the radius. And to the result, the following rule of proportions is applied: If from this the radius is obtained, then how much will be obtained from the radius? This rule of proportion gives rise to the true hypotenuse in a single step. Here, the direct confirmation (*pratyakṣapratīti*) of this is by means of computation carried out on dust spread on the ground or a board (*dhūlīkarma*).

Bhāskara's approximation for the iterated or the *asakṛt-manda-karṇa* is given by

$$K \approx \frac{R^2}{(2R - K_0)}. \tag{9}$$

Here, K_0 is the un-iterated or the initial *manda*-hypotenuse (*sakṛt-manda-karṇa*) which is given by (see eq.(3))

$$K_0 = \left[\{R\sin(\theta_0 - \theta_u)\}^2 + \{R\cos(\theta_0 - \theta_u) + r_0\}^2\right]^{\frac{1}{2}},$$

in terms the mean epicycle radius r_0 and the *manda*-anomaly $(\theta_0 - \theta_u)$. Bhāskara also gives an argument based on the rule of three (*trairāśika*) for the above formula (9), and suggests that it can be confirmed by actual calculation.

It has been noted by K. S. Shukla that Bhāskara's formula (9) for the iterated-*manda*-hypotenuse is only an approximation (it essentially involves assuming $QP \approx QP_1$ in Figure (1)) [MaBk1960, p. 118]. Whatever be the level of accuracy of Bhāskara's formula (9), he was truly a pioneer in his attempt to find an analytical expression, even an approximate one, for the iterated *manda*-hypotenuse. An exact analytical expression for this *aviśiṣṭa-manda-karṇa* was obtained a few centuries later by Saṅgamagrāma Mādhava (c.1340–1425), the legendary founder of the Kerala School, in the form [TaSa2011, pp. 496–497]:

$$K = \frac{R^2}{R_v}, \tag{10}$$

where, R_v is the inverse hypotenuse or *viparīta-karṇa* (OT in Figure (1)), which is given by

$$R_v = \left[R^2 - \{r_0\sin(\theta_0 - \theta_u)\}^2\right]^{\frac{1}{2}} - r_0\cos(\theta_0 - \theta_u). \tag{11}$$

Now, it can easily be seen that Bhāskara's approximate formula (9) can be derived from the Mādhava formula for the iterated-*manda*-hypotenuse given by (10) and (11) in the limit when $\frac{r_0}{R}$ is small.[16] Now, neglecting terms of order $O\left(\left(\frac{r_0}{R}\right)^2\right)$, the non-iterated-*manda*-hypotenuse (*sakṛt-manda-karṇa*), K_0, given by (3) will reduce to

$$K_0 \approx R\left[1 + \left(\frac{r_0}{R}\right)\cos(\theta_0 - \theta_u)\right]. \tag{12}$$

Again, neglecting terms of order $O\left(\left(\frac{r_0}{R}\right)^2\right)$, the *viparīta-karṇa*, R_v, given by (12) reduces to

$$R_v \approx R\left[1 - \left(\frac{r_0}{R}\right) cos(\theta_0 - \theta_u)\right] \approx 2R - K_0, \tag{13}$$

[16] The author is indebted to Prof. M. S. Sriram for kindly communicating this argument to him.

so that the Mādhava formula (10) for the iterated-*manda*-hypotenuse reduces to the approximate expression (9) proposed by Bhāskara.

3.4 The geometrical model of planetary motion according to Bhāskara

Bhāskara discusses the geometrical model of planetary motion in detail in the chapter on graphical representation (*Chedyakādhyāya*) of the *Golādhyāya*. After explaining the graphical representation of Rsines, Bhāskara summarises the graphical representation of planetary motion in verse 7. He explains that, for an observer on the earth, the planetary motion is seen to be non-uniform since the earth is not at the centre of the orbit on which the planets move; and that it is this fact which is crucial for working out the graphical representation of planetary motions [SiŚi1981, p. 386]:

> इदानीं स्पष्टीकरणे फलस्योत्पत्तिमाह –
>
> भूमेर्मध्ये खलु भवलयस्यापि मध्यं यतः स्यात्
> यस्मिन् वृत्ते भ्रमति खचरो नास्य मध्यं कुमध्ये ।
> भूस्थो द्रष्टा न हि भवलये मध्यतुल्यं प्रपश्येत्
> तस्मात् तज्ज्ञैः क्रियत इह तद्दोःफलं मध्यखेटे ॥
>
> यदेतत् भपञ्जरेऽश्विन्यादीनां भानां वलयं तद्भूमेः समन्तात् सर्वत्र तुल्येऽन्तरे वर्तते। यतस्तस्य मध्यं कुमध्ये। अथ यस्मिन् वृत्ते ग्रहो भ्रमति तस्य मध्यं कुमध्ये न। तद्भूमेः समन्तात् समानान्तरं नेत्यर्थः। अतो भूस्थो द्रष्टा भवलये मध्यमस्थाने ग्रहं न पश्यति। किन्त्वन्यत्र पश्यति। तयोर्भवलये यदन्तरं तद्ग्रहस्य फलमित्यर्थादुक्तं भवति। अत उक्तं 'तस्मात् तज्ज्ञः क्रियत इह तद्दोःफलं मध्यखेट' इति।
>
> एवमेकेनैव श्लोकेन संक्षेपाच्छेद्यकसर्वस्वमुक्त्वा ...
>
> *idānīṃ spaṣṭīkaraṇe phalasyotpattimāha —*
>
> *bhūmermadhye khalu bhavalayasyāpi madhyaṃ yataḥ syāt*
> *yasmin vṛtte bhramati khacaro nāsya madhyaṃ kumadhye |*
> *bhūstho draṣṭā na hi bhavalaye madhyatulyaṃ prapaśyet*
> *tasmāt tajjñaiḥ kriyata iha taddoḥphalaṃ madhyakheṭe ||*
>
> *yadetat bhapañjare'śvinyādīnāṃ bhānāṃ valayaṃ tadbhūmeḥ samantāt sarvatra tulye'ntare vartate | yatastasya madhyaṃ kumadhye | atha yasmin vṛtte graho bhramati tasya madhyaṃ kumadhye na | tadbhūmeḥ samantāt samānāntaraṃ netyarthaḥ | ato bhūstho draṣṭā bhavalaye madhyamasthāne grahaṃ na paśyati | kintvanyatra paśyati | tayorbhavalaye yadantaraṃ tadgrahasya phalamityarthāduktaṃ bhavati | ata uktaṃ 'tasmāt tajjñaḥ kriyata iha taddoḥphalaṃ madhyakheṭa' iti |*
>
> *evamekenaiva ślokena saṃkṣepācchedyakasarvasvamuktvā...*
>
> Now he explains the rationale for the corrections in the computation of true planets—

> While the centre of the earth is indeed also the centre of the ring of stars (*bhavalaya*), the centre of the circle in which the moving celestial body (*khacara,* the Sun, Moon and the planets) moves is not located at the centre of the earth. Therefore, the observer on the earth does not see it (celestial body) to have the same (longitude) as the mean on the ring of stars. Hence, those who are knowledgeable of this apply appropriate corrections (*taddoḥphala*), here, to the mean planet (*madhyakheṭa*).
>
> On the celestial sphere (*bhapañjara*), the ring of stars (*bhānāṃ valayam*) comprising of Aśvini etc., lies everywhere uniformly at the same distance from the earth; because its centre is at the centre of the earth. Now, the circle in which the planet (*graha*) moves does not have its centre at the centre of the earth. It means that the circle (in which the planet moves) is not everywhere uniformly at the same distance from the earth. Hence, the observer on the earth does not see the planet as being located at the mean position on the ring of stars. On the other hand, he sees it as being elsewhere. It is thus implied that the difference between the two locations on the ring of stars is the correction for that planet. Therefore it is said that 'Hence, those who are knowledgeable of this apply appropriate corrections to the mean planet' (*tasmāt tajjñaḥ kriyata iha taddoḥphalaṃ madhyakheṭe*).
>
> In this way, having succinctly stated in terms of just a single verse, the whole of graphical representation of planetary motion (*chedyakasarvasva*)...

Thus Bhāskara is emphatic that a central feature of the geometrical picture of planetary motion is that the planets do not move in concentric circles which have their centre located at the centre of the earth. In verses 10–17 Bhāskara describes the eccentric circle (*pratimaṇḍala*) model. In his *vāsanā,* while explaining the method of computing corrections (*phalānayana*) via the eccentric circle model, Bhāskara notes [SiŚi1981, p. 389]:[17]

> कर्णो नाम ग्रहकुमध्ययोरन्तरसूत्रम्। तत् सूत्रं कक्षामण्डले यत्र लग्नं तत्र स्फुटो ग्रहः। स्फुटमध्ययोरन्तरं फलम्।
>
> *karṇo nāma grahakumadhyayorantarasūtram | tat sūtraṃ kakṣāmaṇḍale yatra lagnaṃ tatra sphuṭo grahaḥ | sphuṭamadhyayorantaraṃ phalam |*
>
> The hypotenuse (*karṇa*) is the intervening line joining the planet and the centre of the earth. At the point where that line meets the concentric there is the true planet (*sphuṭagraha*). The [arc] between the true and mean planets [on the concentric] is the correction (*phala*).

Here again, Bhāskara is clearly locating the actual planet (*graha*) on the eccentric. But he also uses the word *sphuṭagraha* to refer to the point where the line along the hypotenuse (*karṇasūtra*), joining the planet and the centre of the concentric, intersects the concentric. This is because the correction

[17] A very similar passage appears, during the discussion of the epicycle model, in the *vāsanā* on verses 28–29: *bhūgrahāntaraṃ karṇaḥ. doḥkoṭivargaikyapadamiti prasiddham. atrāpi prāgvat kakṣāvṛtte karṇasūtrasakte sphuṭagrahaḥ. sphuṭamadhyayorantaraṃ phalamityādi.* [SiŚi1981, p. 390]. A similar passage appears earlier in the *Spaṣṭādhikāra of Grahagaṇitadhyāya* also [SiŚi1981, p. 113].

(*phala*), which is the difference between two longitudes (that of the true and the mean) is always calculated as the arc of a circle. And, here the correction will be the arc on the concentric between the mean planet (which is located on the concentric) and the point where the hypotenuse joining the planet and the centre of the earth intersects the concentric. This use of the word *sphuṭagraha* for the point on the concentric by Bhāskara (and perhaps in some of the works of earlier astronomers also) seems to have caused some confusion, among the traditional as well as modern commentators, in comprehending the geometrical model of planetary motion as presented by Bhāskara.

Amplifying his view that the actual planet is on the eccentric, Bhāskara states in verse 22 that the planet appears smaller while at the apogee (*ucca*) as it is farther; and it appears larger while at perigee (*nīca*) as it is closer [SiŚi1981, p. 389]:

> उच्चस्थितो व्योमचरः सुदूरे नीचस्थितः स्यान्निकटे धरित्र्याः ।
> अतोऽणुबिम्बः पृथुलश्च भाति भानोस्तथासन्नसुदूरवर्ती ॥
>
> *uccasthito vyomacaraḥ sudūre nīcasthitaḥ syānnikaṭe dharitryāḥ |*
> *ato'ṇubimbaḥ pṛthulaśca bhāti bhānostathāsannasudūravartī ||*
>
> When the celestial body is at the apogee it is farther from the earth and when it is at the perigee it is closer to the earth. So, correspondingly its orb (*bimba*) appears minute and large, respectively. Similarly it appears (comparatively) minute (in form and brightness) when closer to the Sun, and large when farther away from the Sun.

Bhāskara discusses the epicycle (*nīcoccavṛtta*) model in verses 23–29 and the equivalence of the eccentric and the epicycle models in verses 31–33. Finally, the geometrical model of planetary motion is presented succinctly in verses 34–35 [SiŚi1981, p. 391]: The centre of the *manda*-epicycle moves on the concentric deferent circle (*kakṣyāvṛtta*); on the *manda*-epicycle (at the location of *manda-sphuṭa*) is located the centre of the *śīghra*-epicycle; the true planet is located on this *śīghra*-epicycle.

> इदानीं मन्दशीघ्रकर्मद्वयेन स्फुटत्वे कारणमाह –
>
> > मध्यगत्या स्वकक्षाख्यवृत्ते व्रजेन्मन्दनीचोच्चवृत्तस्य मध्यं यतः ।
> > तद्धृतौ शीघ्रनीचोच्चमध्यं तथा शीघ्रनीचोच्चवृत्ते स्फुटः खेचरः ॥
> > शीघ्रनीचोच्चवृत्तस्य मध्यस्थितिं ज्ञातुमादौ कृतं कर्म मान्दं ततः ।
> > खेटबोधाय शैघ्र्यं मिथः संश्रिते मान्दशैघ्र्ये हि तेनासकृत् साधिते ॥
>
> नीचोच्चवृत्तभङ्गिपर्यालोचनयैवं परिणमतीति स्पष्टार्थम् ।
>
> *idānīṃ mandaśīghrakarmadvayena sphuṭatve kāraṇamāha —*
>
> > *madhyagatyā svakakṣākhyavṛtte vrajen-*
> > *mandanīcoccavṛttasya madhyaṃ yataḥ |*

tadvṛtau śīghranīcoccamadhyaṃ tathā
śīghranīcoccavṛtte sphuṭaḥ khecaraḥ ||

śīghranīcoccavṛttasya madhyasthitiṃ
jñātumādau kṛtaṃ karma māndaṃ tataḥ |
kheṭabodhāya śaighryaṃ mithaḥ saṃśrite
māndaśaighrye hi tenāsakṛt sādhite ||

nīcoccavṛttabhaṅgiparyālocanayaivaṃ pariṇamatīti spaṣṭārtham |

Now he states why the *manda* and the *śīghra* processes are both needed to determine the true planet:

> The centre of the *manda*-epicycle moves on its concentric with the same rate of motion as the mean planet. On that circle (*manda*-epicycle) is the centre of the *śīghra*-epicycle and similarly the true planet is on the *śīghra*-epicycle. In order to know the location of the centre of the *śīghra*-epicycle, initially the *manda*-process is carried out. Then, to find the true planet, the *śīghra*-process (is carried out). Since the *manda* and *śīghra*-processes are coupled together (*mithaḥ saṃśrite*), they are to be carried out iteratively.

An analysis of the epicycle model shows that this is how the things are settled, and hence the import (of the above verses) is transparent.

Bhāskara does not provide any further explanation in his *vāsanā*. Here, the commentator Nṛsiṃha seems to have misinterpreted 'on that circle' (*tadvṛtau*), in verse 32, as 'on the concentric' (*kakṣāvṛtte*).[18] Thus, according to him the centre of the *śīghra*-epicycle is located in the geocentric concentric at the same longitude as the *manda-sphuṭa* or the *manda*-corrected mean planet. However, even according to Nṛsiṃha, the *śīghra*-corrected true planet is located on the *śīghra*-epicycle and not on the concentric. Earlier, while commenting on Bhāskara's *upapattis* for the revolution numbers (*bhagaṇas*) of the planets, Nṛsiṃha had elaborated his view that the two epicycles *manda* and *śīghra* have their centres on the same geocentric concentric. In that discussion, Nṛsiṃha had even placed the true Sun and the Moon on the concentric, contrary to what Bhāskara has clearly stated in the chapter on lunar eclipses, that the hypotenuse is the radius of the planetary orbit (*karṇo vyāsārdhaṃ grahakakṣyāyāḥ,* cited earlier) and that the true distance of the Sun and the Moon are given by their iterated-*manda*-hypotenuse. Further, Nṛsiṃha had also remarked that, according to the Yavanas, the centre of the *śīghra*-epicycle is located on the *manda*-eccentric and not the geocentric concentric [SiŚi1981, p. 45].

...कक्षावृत्तपरिधौ मन्दनीचोच्चवृत्तमध्यं मध्यगत्या भ्रमति। तत्रोच्चप्रदेशाद्ग्रहो मन्दकेन्द्रगत्या गच्छति। यत्रासौ ग्रहः कर्णसूत्रेण कक्षापरिधौ दृश्यते तत्रैव रविचन्द्रौ स्पष्टौ।

[18] Nṛsiṃha Daivajña's *Vāsanāvārttika* [SiŚi1981, p. 391]: *atra tadvṛtāvityatra kakṣyāvṛtta ityarthaḥ* (the published version has *tadvṛttāvityatra* which is incorrect).

भौमादीनां[19] मन्दस्पष्टस्तत्रैव शीघ्रनीचोच्चवृत्तमध्यं[20] भ्रमति मन्दस्पष्टगत्या तत्परिधौ शीघ्रकेन्द्रगत्यैवोच्चप्रदेशाद्ग्रहो भ्रमति। नीचोच्चवृत्तं नाम अन्त्यफलज्याकृतं वृत्तम्। यवनास्तु मन्दकर्णव्यासार्धेन कृतं यत् कक्षावलयं तत्परिधौ शीघ्रनीचोच्चवृत्तमध्यं मन्यन्ते। एवं मध्यमन्दस्पष्टस्पष्टानां भेदः।

...kakṣāvṛttaparidhau mandanīcoccavṛttamadhyaṃ madhyagatyā bhramati | tatroccapradeśādgraho mandakendragatyā gacchati | yatrāsau grahaḥ karṇasūtreṇa kakṣāparidhau dṛśyate tatraiva ravicandrau spaṣṭau | bhaumādīnāṃ mandaspaṣṭastatraiva śīghranīcoccavṛttamadhyaṃ bhramati mandaspaṣṭagatyā tatparidhau śīghrakendragatyaivoccapradeśādgraho bhramati | nīcoccavṛttaṃ nāmāntyaphalajyākṛtaṃ vṛttam | yavanāstu mandakarṇavyāsārdhena kṛtaṃ yatkakṣāvalayaṃ tatparidhau śīghranīcoccavṛttamadhyaṃ manyante | evaṃ madhyamandaspaṣṭaspaṣṭānāṃ bhedaḥ |

...on the circumference of the concentric, the centre of the *manda*-epicycle moves at the same rate as the mean (planet). In that epicycle, the planet moves from the apogee at the same rate as the *manda*-anomaly. Where this planet is seen on the concentric along the hypotenuse, it is there that the Sun and the Moon are true (*spaṣṭa*). However, for the planets Mars etc., that is the location of *manda-spaṣṭa* where the centre of the *śīghra*-epicycle is also located and it moves with the rate of *manda-spaṣṭa*; the planet moves on that (*śīghra*-epicycle) from the *śīghrocca* at the same rate as the *śīghra*-anomaly. The epicycle is a circle with the same radius as the maximum correction (*antyaphala*). According the Yavanas, the centre of the *śīghra*-epicycle lies on the circumference of that concentric circle which is drawn with the *manda*-hypotenuse as the radius. Thus is the difference between the mean, *manda-spaṣṭa* and *spaṣṭa*.

The geometrical model of Bhāskara seems to have been similarly misinterpreted by Nṛsiṃha's junior contemporary Munīśvara[21] in his commentary *Marīci* on *Siddhāntaśiromaṇi*. Commenting on the above verses 34–35 of the *Chedyakādhikara,* Munīśvara states [SiŚi1943, p. 187]:[22]

ननु तर्हि शीघ्रनीचोच्चवृत्तं कस्मादुल्लेख्यमत आह तद्धृताविति। मन्दकर्णसूत्रसक्तकक्ष्यावृत्तपरिधिप्रदेशे मन्दस्फुटस्थाने।

[19] The published version here reads *bhaumadyastu* which does not make sense.

[20] The published version here reads *śīghroccanīcavṛttamadhye*. Since it does not make sense, we have preferred the variant reading *śīghroccanīcavṛttamadhyaṃ,* given in the same edition. For the same reason, we have also omitted the sentence break at the end of the next word *bhramati.*

[21] Munīśvara or Viśvarūpa (b. 1603), the son of Raṅganātha (author of *Gūḍhārthaprakāśikā* commentary on the *Sūryasiddhānta*) and nephew of Kṛṣṇa Daivajña (author of the *Bījapallava* commentary on Bhāskara's *Bījagaṇita*), hailed from a family of distinguished astronomers who traced their ancestry to Dadhigrāma on Payoṣṇi in Maharashtra and later shifted to Vārāṇāsi. Apart from *Marīci* (written prior to 1638), he wrote several other works including a commentary *Nisṛṣṭārthadūtī* on *Līlāvatī* and an independent *siddhānta* work, *Siddhāntasārvabhauma* (written in 1646) with an auto-commentary (see [CESS, Series A, vol. 4, (1981), pp. 436–441]).

[22] The published version reads '*tadvṛttāviti,*' which is incorrect.

> *nanu tarhi śīghranīcoccavṛttaṃ kasmādullekhyamata āha tadvṛtāviti* | *mandakarṇa-sūtrasaktakakṣyāvṛttaparidhipradeśe mandasphuṭasthāne* |
>
> Then, from where (which centre) should the *śīghra*-epicycle be drawn? So he says 'on that circle' ('*tadvṛtau*'). On that point on the circumference of the concentric where it is in contact with the *manda*-hypotenuse, which is also the location of the *manda-sphuṭa*.

One of the reasons, as to why both Nṛsiṃha and Munīśvara, who were writing nearly five hundred years after the time of Bhāskara, chose to locate the centre of the *śīghra*-epicycle to lie on the concentric, could be that Bhāskara, as noted earlier, himself employed the word *sphuṭagraha* variously for the intersection of the *manda*-hypotenuse and the concentric. It could also be the case that Nṛsiṃha and Munīśvara were perhaps following an earlier tradition of placing both the *manda* and *śīghra* epicycles on the concentric, which is exemplified for instance in the *Vāsanābhāṣya* of Pṛthūdakasvāmin (c. 860) on the *Brāhmasphuṭasiddhānta* of Brahmagupta. Though only portions of this commentary have been published, we do get an idea of Pṛthūdaka's view in a brief summary of the geometrical model of planetary motion that he presents in his commentary on chapter XXI (*Golādhyāya*) of the work which has been published. Commenting on verse 29 of chapter XXI (which has been cited earlier, in connection with Brahmagupta's explanation as to why the *manda*-hypotenuse does not play any role in the *manda*-correction), Pṛthūdaka notes [BSS2003, p. 103]:

> भौमादीनां पुनर्मन्दकर्मणा यः प्रदेशः सिद्धो भवति कक्षामण्डले तत्र शीघ्रनीचोच्चवृत्तमध्यं कृत्वा शेषं प्रदर्शयेत्।
>
> *bhaumādīnāṃ punarmandakarmaṇā yaḥ pradeśaḥ siddho bhavati kakṣāmaṇḍale tatra śīghranīcoccavṛttamadhyaṃ kṛtvā śeṣaṃ pradarśayet* |
>
> In the case of (the planets) Mars etc., again the location on the concentric circle which is obtained by the process of *manda*-correction (and has the longitude of *manda-sphuṭa*), there the centre of *śīghra*-epicycle is to be located, and the rest is to be demonstrated.

As was noted earlier, Pṛthūdaka's view that there is no necessity for introducing the iterated-*manda-karṇa,* expressed while commenting on the same verse, has been summarily rejected by Bhāskara.

Notwithstanding the views of Nṛsiṃha and Munīśvara or even the tradition that seems to go back to Pṛthūdaka, it is quite clear, from what we have discussed earlier, that Bhāskara locates the real planet (obtained after the correction) on the epicycle or the eccentric and not on the concentric. That is why he repeatedly emphasises that the hypotenuse is the intervening line joining the planet and the centre of the earth ('*karṇo nāma grahakumadhyay-orantara sūtram*'). In the case of the Sun or the Moon, since there is only

the *manda*-correction, the *manda-sphuṭa* or the true Sun or Moon, is unambiguously located by him on the *manda*-epicycle. In the same way, he also prescribes that the iterated-*manda*-hypotenuse should be taken as the true distance between the centre of the earth and the Sun or the Moon. Therefore, it is but natural that, in the case of the five planets Mars etc., also, Bhāskara would locate *manda-sphuṭa* on the epicycle. Clearly, the interpretation of the above passage of *Siddhāntaśiromaṇi,* as given by Nṛsiṃha and Munīśvara, that the centre of the *śīghra*-epicycle is at the *manda-sphuṭa* which is located on the concentric is wrong, even if they were following some tradition, going back to Pṛthūdaka, on this issue.

Thus, the geometrical model of planetary motion, according to Bhāskara, is that the Sun and the Moon move on the *manda*-epicycle, whose radius varies with the variable *manda*-hypotenuse. In the case of the five planets Mars etc., this is the circle on which the *manda-sphuṭa* is located. The planets themselves move in the *śīghra*-epicycle, whose centre is located at the *manda-sphuṭa*.

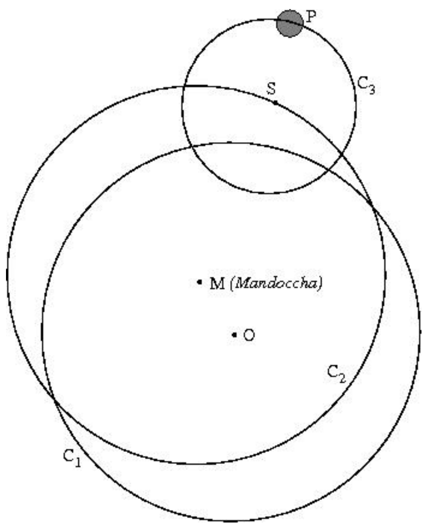

Figure 2: Geometrical model of planetary motion.

This epicycle-epicycle model (or equivalently, eccentric-epicycle model as shown in Figure (2)) of planetary motion is what is later discussed in detail by the Kerala School astronomers Parameśvara and Nīlakaṇṭha Somayājī [TaSa2011, pp. 505–523]. Of course, Nīlakaṇṭha also proposed a major revision of the traditional model by abandoning the view that the mean Mercury and mean Venus should be identified with the mean Sun.

4 Bhāskara on the precession of equinoxes

While the retrograde motion of the equinoxes and solstices (*ayanacalana*) was known in the Indian astronomical tradition, at least from the time of *Varāhamihira* (c. 550), there were different views as to whether the motion was in the nature of precession over the entire ecliptic, or of trepidation or oscillation over some arc of it [Pin1972]. In verses 17–19 of *Golabandhādhikāra* of the *Golādhyāya,* Bhāskara discusses the precession of the equinoxes. He mentions that according to the ancient tradition of *Sūryasiddhānta* the equinoxes undergo 30,000 retrograde revolutions (*vyastā bhagaṇāḥ*) in a *kalpa* of 4.32×10^9 years. He then points out this is the same phenomenon of precession of equinoxes which has been discussed by Muñjāla and others, according to whom the equinoxes undergo 199,669 retrograde revolutions in a *kalpa.* He further notes that the amount of precession has to be added to the longitude of the planet's longitude in order to compute its declination etc. [SiŚi1981, p. 397]:[23]

> विषुवत्क्रान्तिवलययोः संपातः क्रान्तिपातः स्यात् ।
> तद्भगणाः सौरोक्ताः व्यस्ता अयुतत्रयं कल्पे ॥
> अयनचलनं यदुक्तं मुञ्जालाद्यैः स एवायम् ।
> तत्पक्षे तद्भगणाः कल्पे गोऽङ्गर्त्तुनन्दगोचन्द्राः ।
> तत्सञ्जातं पातं क्षिप्त्वा खेटेऽपमः साध्यः ।
>
> *viṣuvatkrāntivalayayoḥ saṃpātaḥ krāntipātaḥ syāt |*
> *tadbhagaṇāḥ sauroktā vyastā ayutatrayaṃ kalpe ||*
> *ayanacalanaṃ yaduktaṃ muñjālādyaiḥ sa evāyam |*
> *tatpakṣe tadbhagaṇāḥ kalpe go'ṅgarttunandagocandrāḥ |*
> *tatsaṃjātaṃ pātaṃ kṣiptvā kheṭe'pamaḥ sādhyaḥ ||*
>
> The equinoxes (*krāntipāta*) are the [points of] intersection of the celestial equator and the ecliptic. Their retrograde revolutions are mentioned to be 30,000 in a *kalpa* in the *Sūryasiddhānta.* This is the motion of the solstices (*ayanacalana*) stated by Muñjāla and others. In their school, the revolutions of them (the equinoxes) in a *kalpa* are given as 1,99,669. Adding the amount of precession obtained this way to [the longitude of] the planet, the declination (*apama*) is to be found.

In his *vāsanā,* Bhāskara begins by highlighting the fact that the precession of equinoxes is an observed phenomenon, though it was not accepted by Brahmagupta [BSS1902, p. 168].[24] and others as the effect (or the displacement of the equinox from the beginning of *Meṣa* (*Meṣādi*)) was indeed small at that

[23] Earlier, in the *Grahagaṇitādhyāya,* Bhāskara refers to the amount of precession in degrees (*ayanāṃśa*) in verse 2 of *Pātādhikāra.* In his *vāsanā* on this verse, Bhāskara also explains how the *ayanāṃśa* may be determined [SiŚi1981, p. 308]. The author is grateful to Prof. M. S. Sriram for bringing this to his attention.

[24] In verse 54 of Chapter XI (*Tantraparīkṣādhyāya*), Brahmagupta states that the two solstices are stationary ('*sthiram ayanadvayam*').

time. He also draws attention to the fact that the phenomenon was mentioned in the ancient tradition (*āgama*) of the *Sūryasiddhānta*. As regards the justification for conceiving such a movement, he draws a parallel with the way in which the movement of the apsides and the nodes are inferred from the observed motion of the lunar apogee and nodes.

Bhāskara then raises the issue that, in the case of a phenomenon such as the precession of the equinoxes which becomes appreciable only over a long period of time, it is possible that different revolution numbers (*bhagaṇas*) could still lead to the same results which are in agreement with contemporary observations. One could therefore use any suitable rate of precession that is in accordance with observations. Emphasising the role of continuing tradition in astronomy Bhāskara states that when, over time, the precession effects become substantial, there would be future Brahmaguptas who would compose treatises after firmly establishing the rate of precession which is in accordance with observations. It is only thus, by being sustained by such great savants, that the science of astronomy continues to retain accuracy over very long periods.

Finally, Bhāskara states that the number of revolutions of the equinoxes in a *kalpa* could be 30,000 as mentioned in the *Sūryasiddhānta,* or they could be 1,99,669 as stated by Muñjāla and others. At any given time, whatever be the actual displacement of the equinox as determined by proficient astronomers, that is what should be used in calculating the declination, etc. [SiŚi1981, p. 398]:

क्रान्त्यर्थं पातः क्रान्तिपातः। पातो नाम संपातः। कयोः। विषुवत्क्रान्तिवलययोः। नहि तयोर्मेषादावेव संपातः किन्तु तस्यापि चलनमस्ति। येऽयनचलनभागाः प्रसिद्धाः त एव विलोमगस्य क्रान्तिपातस्य भागाः। मेषादेः पृष्ठतस्तावद्भागान्तरे क्रान्तिवृत्ते विषुवद्वृत्तं लग्नमित्यर्थः। नहि क्रान्तिपातो नास्तीति वक्तुं शक्यते। प्रत्यक्षेण तस्योपलब्धत्वात्। उपलब्धिप्रकारमग्रे वक्ष्यति।

तत् कथं ब्रह्मगुप्तादिभिर्निपुणैरपि नोक्त इति चेत्। तदा स्वल्पत्वात् तैर्नोपलब्धः। इदानीं बहुत्वात् साम्प्रतिकैरुपलब्धः। अत एव तस्य गतिरस्तीत्यवगतम्। यद्येवमनुपलब्धोऽपि सौरसिद्धान्तोक्तत्वादागमप्रामाण्येन भगणपरिध्यादिवत् कथं तैर्नोक्तः। सत्यम्। अत्र गणितस्कन्ध उपपत्तिमानेवागमः प्रमाणम्। तर्हि मन्दोच्चपातभगणाः आगमप्रामाण्येनैव कथं तैरुक्ता इति न च वक्तव्यम्। यतो ग्रहाणां मन्दफलाभावस्थानानि प्रत्यक्षेणैवोपलभ्यन्ते। तान्येव मन्दोच्चस्थानानि। यान्येव विक्षेपाभावस्थानानि तान्येव पातस्थानानि। किन्तु तेषां गतिरस्ति नास्ति वेति सन्दिग्धम्। तत्र मन्दोच्चपातानां गतिरस्ति चन्द्रमन्दोच्चपातवदित्यनुमानेन सिद्धा।

सा च कियती। तदुच्यते। यैर्भगणैरुपलब्धिस्थानानि तानि गणितेनागच्छन्ति तद्भगणसंभवा वार्षिकी दैनन्दिनी वा गतिर्ज्ञेया। नन्वेवं यद्यन्यैरपि भगणैस्तान्येव स्थानान्यागच्छन्ति तदा कतरस्या गतेः प्रामाण्यम्। सत्यम्। तर्हि साम्प्रतिकोपलब्ध्यनुसारिणी कापि गतिरङ्गीकर्तव्या।

यदा पुनर्महता कालेन महदन्तरं[25] भविष्यति तदा महामतिमन्तो ब्रह्मगुप्तादीनां समानधर्माणः ये उत्पत्स्यन्ते ते[26] तदुपलब्ध्यनुसारिणीं गतिमुररीकृत्य शास्त्राणि करिष्यन्ति। अत एवायं गणितस्कन्धो महामतिमद्भिर्धृतः सन् अनाद्यन्तेऽपि काले खिलत्वं न याति।

अतोऽस्य क्रान्तिपातस्य भगणाः कल्पेऽयुतत्रयं तावत् सूर्यसिद्धान्तोक्ताः । तथा मुञ्जालाद्यैर्यदयनचलनमुक्तं स एवायं क्रान्तिपातः। ते गोऽङ्गर्त्तुनन्दगोचन्द्रा १९९६६९ उत्पद्यन्ते। अथ च ये वा ते वा भगणा भवन्तु। यदा येंऽशाः निपुणैरुपलभ्यन्ते तदा स एव क्रान्तिपात इत्यर्थः। तं विलोमगं क्रान्तिपातं ग्रहे प्रक्षिप्य क्रान्तिः साध्या।

krāntyartham pātaḥ krāntipātaḥ | pāto nāma saṃpātaḥ | kayoḥ | viṣuvatkrāntivalayayoḥ | nahi tayormeṣādāveva saṃpātaḥ kintu tasyāpi calanamasti | ye'yanacalanabhāgāḥ prasiddhāsta eva vilomagasya krāntipātasya bhāgāḥ | meṣādeḥ pṛṣṭhatastāvadbhāgāntare krāntivṛtte viṣuvadvṛttaṃ lagnamityarthaḥ | nahi krāntipāto nāstīti vaktuṃ śakyate | pratyakṣeṇa tasyopalabdhatvāt | upalabdhiprakāramagre vakṣyati |

tat kathaṃ brahmaguptādibhirnipuṇairapi nokta iti cet | tadā svalpatvāt tairnopalabdhaḥ | idānīṃ bahutvāt sāmpratikairupalabdhaḥ | ata eva tasya gatirastītyavagatam | yadyevamanupalabdho'pi saurasiddhāntoktatvādāgamaprāmāṇyena bhagaṇaparidhyādivat kathaṃ tairnoktaḥ | satyam | atra gaṇitaskandha upapattimānevāgamaḥ pramāṇam | tarhi mandocapātabhagaṇāḥ āgamaprāmāṇyenaiva kathaṃ tairuktā iti na ca vaktavyam | yato grahāṇāṃ mandaphalābhāvasthānāni pratyakṣeṇaivopalabhyante | tānyeva mandoccasthānāni | yānyeva vikṣepābhāvasthānāni tānyeva pātasthānāni | kintu teṣāṃ gatirasti nāsti veti sandigdham | tatra mandoccapātānāṃ gatirasti candramandoccapātavadityanumānena siddhā | sā ca kiyatī | taducyate | yairbhagaṇairupalabdhisthānāni tāni gaṇitenāgacchanti tadbhagaṇasaṃbhavā vārṣikī dainandinī vā gatirjñeyā | nanvevaṃ yadyanyairapi bhagaṇaistānyeva sthānānyāgacchanti tadā katarasyā gateḥ prāmāṇyam | satyam | tarhi sāmpratikopalabdhyanusāriṇī kāpi gatiraṅgīkartavyā | yadā punarmahatā kālena mahadantaraṃ bhaviṣyati tadā mahāmatimanto brahmaguptādīnāṃ samānadharmāṇaḥ ye utpatsyante te tadupalabdhyanusāriṇīṃ gatimurarīkṛtya śāstrāṇi kariṣyanti | ata evāyaṃ gaṇitaskandho mahāmatimadbhirdhṛtaḥ sannanādyante'pi kāle khilatvaṃ na yāti |

ato'sya krāntipātasya bhagaṇāḥ kalpe'yutatrayaṃ tāvat sūryasiddhāntoktāḥ | tathā muñjālādyairyadayanacalanamuktaṃ sa evāyaṃ krāntipātaḥ | te go'ṅgarttunandagocandrā 199669 utpadyante | atha ca ye vā te vā bhagaṇā bhavantu | yadā ye'ṃśāḥ nipuṇairupalabhyante tadā sa eva krāntipāta ityarthaḥ | taṃ vilomagaṃ krāntipātaṃ grahe prakṣipya krāntiḥ sādhyā |

The points of intersection that are relevant for computing the declination are the equinoxes (*krāntipāta*). *Pāta* refers to a meeting point. [Meeting point] of which entities? Of the celestial equator, and the ecliptic. It is not the case that they intersect only at the beginning of *Meśa,* since that (point of intersection) also has a motion. The well known amount of *ayanacalana* in degrees, is actually the amount in degrees of (the displacement of) the equinoxes which have a retrograde motion. It only means that the celestial equator meets the ecliptic at a distance of so many degrees behind (to the west of) the beginning of *Meśa.* It cannot be said that there is no equinox, since it cannot be directly observed. The method

[25] Here we have adopted the reading '*mahadantaram*' from [SiŚi1943, p. 210], which is better than the reading '*mahantaram*' found in [SiŚi1981, p. 398].

[26] The published editions [SiŚi1981, p. 398], [SiŚi1943, p. 210] here read '*samānadharmāṇa evotpatsyante. te*'. This seems to be a misreading as *eva* is not in the appropriate place.

> of observation will be stated later. It may then be asked as to why that (motion of the equinox) has not been mentioned even by proficient astronomers such as Brahmagupta and others. Since it (the amount) was small at that time, it was not observed. Now, since it is considerable, it has been observed by the contemporary astronomers. Hence, it has been understood that there is a movement of equinox. Even if it were not observed, by being mentioned in the *Sūryasiddhānta,* it (the movement of equinoxes) has been validated by reliable tradition (*āgama*). Still, why did they (Brahmagupta and others) not mention it, in the same manner as they mentioned the various revolution numbers and the circumferences of epicycles (which were also based on the authority of tradition)? True. Here, in the computational branch (*gaṇitaskandha*) [of astral sciences], only an authoritative tradition which is supported by justification is a valid source of knowledge (*upapatimān āgama eva pramāṇam*). But, then you should not say that how did they (Brahmagupta and others) present the motions of the apsides and nodes merely by the authority of tradition? After all, the locations where the planets do not have a *manda*-correction are obtained by observation itself. Those are the locations of the *mandoccas.* The places where there is no latitudinal deflection, those are the locations of the nodes. However, it is doubtful as to whether they (apsides and nodes) have any movement or not. There, it should be clear that the apsides and nodes have a movement; because, that is established by inference, following the fact that the lunar apogee and node [are seen to have such a movement].
>
> Now we shall state how much is that [rate of motion of the equinoxes]. We should adopt only such annual or daily rates of motion, that result from the revolution numbers which lead, via computation, to the same locations that are in accordance with observations. If such is the case, then which rate of motion is to be taken as valid when different revolution numbers lead to the same observed results? True. Then we may accept any one of the rates which gives results in concordance with current observations. If, after a long period of time, there arises a large difference (between computation and observation), then the highly intelligent astronomers of the same class as Brahmagupta, who arise, they will firmly establish the rates which are in concordance with the observations and prepare treatises. It is only because of this that this mathematical branch [of astral sciences], being sustained by highly intelligent astronomers, does not become inaccurate, even over times which have no beginning or end (*anādyante'pi kāle*).
>
> Now, the number of revolutions of the equinox have been said to be 30,000 in a *kalpa* in the *Sūryasiddhānta.* On the other hand, the motion of the solstices (*ayanacalana*), stated by Muñjāla and others, is also about [the motion of the] the same equinoxes. They (the revolution numbers given by Muñjāla and others) are 1,99,669. Now, let the revolution numbers be either these (given by Muñjāla and others) or those (given by the *Sūryasiddhānta*). Whatever be the number of degrees obtained by experts [for the displacement of the equinoxes] at a given time, that is then taken to be [the correct location of] the equinoxes. Adding [the displacement of] the equinoxes which have a retrograde motion, to the longitude of the planet, the declination is to be computed.

While, in the above passages, Bhāskara presents both the tradition of *Sūryasiddhānta* and the view of Muñjāla on the rate of precession, he clearly adopted the rate given by Muñjāla in his later manual *Karaṇakutūhala* (c. 1183) [KaKu1991, p. 28]:

> अथायनांशाः करणाब्दलिप्ताः युक्ता भवास्तद्युतमध्यभानोः ।
>
> *athāyanāṃśāḥ karaṇābdaliptā yuktā bhavāstadyutamadhyabhānoḥ* |
>
> Then, the amount of precession (*ayanāṃśa*) is given by number of minutes equal to the years elapsed since the epoch of the manual (*karaṇābda*) together with 11 degrees...

From the above prescription, it follows that the annual rate of precession is given as $1'$; and the amount of precession at the time of the epoch of the work, namely *Śakābda* 1105 (1183 CE), is given as $11°$. The annual rate of precession corresponding to 1,99,669 revolutions in a *kalpa* of 4.32×10^9 years, corresponds to an annual rate of around $59''54'''$ and the amount of precession from the beginning of the *kalpa* works out to be around $10°54'35''$. These have been approximated in the above prescription of *Karaṇakutūhala,* as has also been noted by Munīśvara in his commentary *Marīci* on *Siddhāntaśiromaṇi* (see below).

4.1 Bhāskara's citations from the earlier works regarding the movement of equinoxes

Bhāskara's discussion of precession of the equinoxes has been critiqued by David Pingree who has especially questioned the veracity of Bhāskara's citations from *Sūryasiddhānta* and Muñjāla. In his scholarly overview of 'Precession and Trepidation in Indian Astronomy Before A. D. 1200', Pingree remarks [Pin1972, pp. 32–33]:

> Finally, Bhāskara (A. D. 1150 has a very difficult passage (*Siddhāntaśiromaṇi, Golādhyāya* 7, 17–18):
>
> "The intersection of the equinoctial and the declinational circles is 'the node of declination'. Its retrograde revolutions are said by the Saura [*Sūryasiddhānta*] to be 30,000 in a *Kalpa.* But the precessional motion proclaimed by Muñjāla and so on is correct; in this school its revolutions in a *Kalpa* are 199,699."
>
> Bhāskara's reference here to the *Sūryasiddhānta* has caused much learned comment as it clearly conflicts with the verse from that work (3,9) cited as early as Govindasvāmin; it also conflicts with the statement regarding Sūrya's theory of precession made by Viṣṇucandra. The expression of 600 in *Sūryasiddhānta* is '*triṃśatkṛtyo*' (thirty twenties); in some versions, the word for twenties, 'kṛtyo', is corrupt. Perhaps, Bhāskara's statement reflects someone's misunderstanding of the *Sūryasiddhānta,* which was interpreted as saying that there are thirty trepidations in a *Mahāyuga* only, and therefore 30,000 in a *Kalpa.*
>
> Bhāskara's statement regarding Muñjāla is not without difficulties either, though the number of revolutions in a *Kalpa* is closer to 200,000 than any other that is attested. Muñjāla, as was shown above, is known to have followed Govindasvāmin's

second theory. Unless he wrote a third work besides the *Bṛhanmānasa* and the *Laghumānasa,* Bhāskara's reference must be wrong.

It is not clear why Pingree characterises verses 17–18 of the *Golabandhādhikāra* of the *Golādhyāya* cited above as 'a very difficult passage'. There is indeed no difficulty in comprehending what Bhāskara has stated, either in these verses or his commentary thereon, as far as mathematical astronomy is concerned. Clearly his view is that there is a precession of equinoxes, that different rates of precession have been mentioned in earlier texts, and that a proficient astronomer should choose the rate that is in accordance with contemporary observations. He himself has preferred to adopt the rate mentioned by Muñjāla.

Pingree's criticism is perhaps only addressed to the veracity of the citations made by Bhāskara from the *Sūryasiddhānta* and from Muñjāla.[27] Now, the currently available version of *Sūryasiddhānta* (which has been dated to 8th century CE by Pingree) has the following verse [SūSi1891, p. 98]:

त्रिंशत्कृत्यो युगे भानां चक्रं प्राक् परिलम्बते।

triṃśatkṛtyo yuge bhānāṃ cakraṃ prāk parilambate |

In a (*mahā*) *yuga* the ecliptic extends six hundred (thirty-twenties) times eastwards.

All the commentators have interpreted the above passage as implying that the equinox has an oscillatory motion (trepidation) moving 27° either side of the *Meśādi,* 600 times in a *mahāyuga* of 43,20,000. This means that 108° are covered in a period of 7,200 years, thereby implying an annual rate of movement of 54″.

From the fact that Bhāskara, in his *Vāsanābhāṣya,* refers to the notion of movement of equinoxes given in *Sūryasiddhānta* as a tradition (*āgama*) that was known to Brahmagupta and others, it should be clear that he is referring to an ancient version of *Sūryasiddhānta.* That such versions existed is clear from the summary available in *Pañcasiddhāntikā* of Varāhamihira (c. 550), and Bhāskara could be referring to one such version.[28] However, if the number of

[27] The same issue was discussed, almost two hundred years earlier, in a pioneering study of Indian astronomy by Henry Thomas Colebrooke [Col1816]. Colebrooke also talks of the 'difficulties' presented by the passage of Bhāskara to the historian of astronomy. Colebrooke, however, concludes that Bhāskara was following an authentic tradition of Muñjāla, based on the reference to Muñjāla found in the commentary *Marīci* of Munīśvara (see below).

[28] In his commentary on *Brāhmasphuṭasiddhānta*, Sudhākara Dvivedi cites a verse of Viṣṇucandra (c. 550 CE), author of a *Vasiṣṭhasiddhānta,* which mentions that the number of revolutions of the solstices in a (*mahā*)*yuga* are 1,89,411 and that 'this was formerly the opinion of Brahma, Sūrya, etc.', '*brahmārkādi matam purā'* [BSS1902, p. 168]. This verse of Viṣṇucandra (which has also been cited by Colebrooke [Col1816, pp. 214–215], and

revolutions of the equinox in a *kalpa* are 30,000 (*ayutatraya*), as mentioned in the above verse of *Siddhāntaśiromaṇi,* then the rate of motion of the equinox will turn out to be 9″ per annum, which is too small. To be fair, we should also note that Bhāskara is not advocating this rate precession as the accurate one, he is mainly citing this as an ancient authority (*āgama*) for the movement of equinoxes. In this context, it may also be noted that Nṛsiṃha Daivajña in his *Vāsanāvārttika* has suggested that the word '*ayutatraya*' (30,000) is perhaps an erroneous reading of the original, which should have been '*niyutatraya*' (300,000) [SiŚi1981, p. 399].[29] If that reading were to be accepted, then the annual rate of precession will work out to be a more reasonable figure of 40″.

As regards Bhāskara's citation from Muñjāla, it may be noted that the motion of equinoxes is mentioned right at the beginning of his work *Laghumānasa.* In the section *Dhruvakanirūpaṇādhikāra* which gives the initial values and other parameters useful for making actual computations, and is usually tagged on with verse 2 of the text in various commentaries, Muñjāla gives 6°50′ as the amount of displacement of the equinoxes from the *Meśādi* for *śakābda* 854 (932 CE), and 1′ per year as the rate of motion of the equinox, without mentioning whether the movement is in the nature of precession or trepidation:[30]

> अयनचलनाष्षडंशाः पञ्चाशल्लिप्तिकास्तथैकैकाः ।
> प्रत्यब्दं तत्सहितो रविरुत्तरविषुवदादिः स्यात् ॥
>
> *ayanacalanāṣṣaḍaṃśāḥ pañcāśalliptikāstathaikaikāḥ |*
> *pratyabdaṃ tatsahito raviruttaraviṣuvadādiḥ syāt ||*
>
> The amount of displacement of the solstices (*ayanacalana*) is 6°50′ and it increases by 1′ each year. The (longitude of the) Sun increased by that amount will give the (longitude) measured from the vernal equinox (*uttara-viṣuvat*).

In fact, the relevant verses of Muñjāla on the precession of equinoxes, which perhaps constituted the source for Bhāskara also, have been cited by Munīśvara, who attributes the verses to Muñjāla without however naming the work from which they have been cited. In his commentary Marīci on the above verses of the *Golabandhādhikāra* of *Golādhyāya,* Munīśvara first cites three

Pingree [Pin1972, p. 32] clearly shows that there was an earlier version of *Sūryasiddhānta,* which talked of precession and not the trepidation of the equinoxes.

[29] '*ayutatrayaṃ kalpa ityatra niyutaśabdasyalakṣasaṅkhyāvācakatvena niyutatrayaṃ kalpa iti vā pāṭhaḥ sādhīyān*'. Such a variant reading has also been suggested by Munīśvara as a possibility: '*ayutetyatra niyutetipāṭhastena niyutaśabdasya lakṣasaṅkhyāvācakatvād-vyastāḥ paścimā bhagaṇā lakṣatrayam'* [SiŚi1943, p. 214].

[30] [LaMa1944, pp. 4–5]. Also, [LaMa1952, p. 3]. Also, [LaMa1990, p. 62]. N. K. Majumdar suggests that these verses could be from *Bṛhanmānasa,* as they have been cited separately from the text by the commentator Praśastidhara. In any case, these verses are an essential part of *Laghumānasa,* and their role is the same as that of the verses of the *Daśagītikā in Āryabhaṭīya.*

and a half verses of Muñjāla on the precession of equinoxes,[31] and goes on to cite another half verse a little later in his commentary [SiŚi1943, pp. 216–217]:

> मुञ्जालादिमते 'तद्भगणाः' क्रान्तिपातभगणा व्यस्ताः ब्रह्मदिने ।
> एकत्रिंशदधिकशतत्रयोनं लक्षद्वयम् ।
> 'तद्भगणाः कल्पे स्युर्गोरसरसगोऽङ्कचन्द्रमिताः ॥'
>
> इति तद्वचनादुक्ताः। एतदनुरोधेन प्रतिवर्षं तद्गतिः ... प्रत्यक्षसंवादासन्नं ५९/५४/२/३१/१२। अत एवाऽऽचार्यैः करणकुतूहल एतद्भगणेभ्यस्तत्कालेऽयनग्रहं भगणाद्यमेनमानीय...स्वल्पान्तरेण एकादशांशानङ्गीकृत्य प्रतिवर्षं तद्गतिकलां चाङ्गीकृत्य 'अयनांशाः करणादेर्लिप्ता युक्ता भवा' इत्ययनांशसाधनं निबद्धम्।
>
> *muñjālādimate 'tadbhagaṇāḥ' krāntipātabhagaṇā vyastāḥ brahmadine |*
> *ekatriṃśadadhikaśatatrayonaṃ lakṣadvayam |*
> *'tadbhagaṇāḥ kalpe syurgorasarasago'ṅkacandramitāḥ ||'*
>
> *iti tadvacanāduktāḥ | etadanurodhena prativarṣaṃ tadgatiḥ ...pratyakṣasaṃvādāsannaṃ 59/54/2/31/12 | ata evā"cāryaiḥ karaṇakutūhala etadbhagaṇebhyastatkāle'yanagrahaṃ bhagaṇādyamenamānīya...svalpāntareṇa ekādaśāṃśānaṅgīkṛtya prativarṣaṃ tadgatikalāṃ cāṅgīkṛtya 'ayanāṃśāḥ karaṇāderliptā yuktā bhavā' ityayanāṃśasādhanaṃ nibaddham |*
>
> According to Muñjāla and others, 'its revolutions' [namely], the number of retrograde revolutions of the equinoxes in a *kalpa* (a day of the Brahmā), which are 1,99,669 (2,00,000-331).
>
> 'Its number of revolutions in a *kalpa* are determined to be 1,99,669' (*tadbhagaṇāḥ kalpe syurgorasarasago'ṅkacandramitāḥ*),
>
> as have been stated in his (Muñjāla's) words. Following this, every year its motion will be ...as obtained also by tallying with observations, nearly $59''54^{(''')}2^{(iv)}31^{(v)}12^{(vi)}$. It is only because of this that, the Ācārya has made use of these revolution numbers in *Karaṇakutūhala* to find, in his time, the extent of displacement of the equinox (*ayanagraha*), in revolutions, etc., ...and adopted a value very close to the result, namely 11°; he has also adopted the rate of annual motion of that (the equinoxes) to be 1′, and has prescribed the following rule for determining the amount of precession of equinoxes (*ayanāṃśa*): 'The amount of precession is given by number of minutes equal to the years elapsed since the epoch of the *karaṇa* added to 11 degrees'.

Surprisingly, Pingree does not make any reference to the above verses of Muñjāla cited by Munīśvara, even though he has taken note of certain other references made by Munīśvara (to the *Sūryasiddhānta* and to the views of Āryabhaṭa II) during the course of the commentary on the same verses of Bhāskara [Pin1972, pp. 34–35]. Later scholars have suggested that these verses of Muñjāla, as cited by Munīśvara, are from his other work *Bṛhanmānasa*. (See [SiŚi1981, p. 399] and [LaMa1990, p. 5]). This work of Muñjāla has been mentioned by various commentators of *Laghumānasa* and also by al Biruni

[31] '*muñjālādyairācāryaiḥ. "yaduttaratoyāmyadiśaṃ yāmyātastadanu saumyadigbhāgam....nirdiṣṭo'yanasandhiścalanaṃ tatraiva saṃbhavati". iti anenāyanacalanaṃ kalpādau ...*' [SiŚi1943, p. 216].

[AIB1910, vol. I, p. 157], but so far no manuscripts have been found. However, Pingree's claim that, as regards Muñjāla, 'Bhāskara's reference must be wrong', seems totally unjustified.'

5 Bhāskara on the problem of the latitude of interior planets

The planetary models of classical Indian astronomy (like their counterparts in other traditions), were clearly off the mark in prescribing that the *manda*-correction or the equation of centre for the interior planets, Mercury and Venus, should be applied to the mean Sun. However, at least from the time of *Āryabhaṭa,* the Indian astronomers (unlike their counterparts in Greco-European and Islamic traditions prior to Kepler) were able to compute the latitudinal motions of these planets fairly correctly by making use of the notion of their *śīghroccas,* which were essentially the mean heliocentric planets. The standard procedure for computing the latitude of a planet involves the Rsine of the latitudinal anomaly (*kṣepakendra*), which is actually the difference between the mean longitude of the planet (corrected by the equation of centre) and the node. For the exterior planets, the latitudinal anomaly was correctly prescribed as the sum of the *manda-sphuṭa* and the computed node (*pāta*), where the sum was taken since the motion of nodes is retrograde. However, in the case of the interior planets, a fairly accurate formulation was arrived at by the prescription that their latitudinal anomaly should be taken as the sum of the *śīghrocca* (as that was actually the mean heliocentric planet) and the node.

This, however, led to the paradox that there were two different rules for computing the latitudes of planets: They were to be computed from the *manda-sphuṭa* for the exterior planets, and from the *śīghrocca* for the interior ones. While trying to explain the latitudinal motion in the *Golabandhādhikāra* of *Golādhyāya,* Bhāskara is acutely aware of this duality. However, the solution he proposes is somewhat naive, namely that the revolution numbers of the nodes of Mercury and Venus which have been tabulated (in *Siddhāntaśiromaṇi* as well as other texts) are actually true revolution numbers of the nodes diminished by those of their *śīghra*-anomaly (which is *śīghrocca* minus the *manda-sphuṭa*). In this way, Bhāskara is able to trivially solve the problem of duality and claim that the procedure for computing the latitudes of Mercury and Venus also, in actuality, involves only their *manda-sphuṭas,* as in the case of the exterior planets.

After discussing the general procedure for computing the latitudes of planets in verses 20–22 of *Golabandhādhikāra,* Bhāskara addresses the exceptional case of Mercury and Venus in verse 23 [SiŚi1981, pp. 402–403]:

इदानीं ज्ञशुक्रयोर्विशेषमाह –

ये चात्र पातभगणाः पठिता ज्ञभृग्वोः ते शीघ्रकेन्द्रभगणैरधिका यतः स्युः ।
स्वल्पाः सुखार्थमुदिताश्चलकेन्द्रयुक्तौ पातौ तयोः पठितचक्रभवौ विधेयौ ॥

... ननु ज्ञशुक्रयोः शीघ्रोच्चपातयुतिं केन्द्रं कृत्वा यो विक्षेप आनीतः स शीघ्रोच्चस्थान एव भवितुमर्हति न ग्रहस्थाने। यतो ग्रहोऽन्यत्र वर्तते। अत इदमनुपपन्नमिव प्रतिभाति।

तथा च ब्रह्मसिद्धान्तभाष्ये 'ज्ञशुक्रयोः शीघ्रोच्चस्थाने यावान् विक्षेपस्तावानेव यत्र तत्रस्थस्यापि ग्रहस्य भवति। अत्रोपलब्धिरेव वासना नान्यत् कारणं वक्तुं शक्यत' इति चतुर्वेदेनाप्यध्यवसायोऽत्र कृतः।

सत्यम्। अत्रोच्यते। येऽत्र ज्ञशुक्रयोः पातभगणाः पठितास्ते शीघ्रकेन्द्रभगणैर्युताः सन्तः तद्भगणा भवन्ति। तथा च माधवीये सिद्धान्तचूडामणौ पठिताः। अतोऽल्पभगणभवः पातः स्वशीघ्रकेन्द्रेण युतः कार्यः। शीघ्रोच्चाद्ग्रहे शोधिते शीघ्रकेन्द्रम्। तस्मिन् सपाते क्षेपकेन्द्रकरणार्थं ग्रहः क्षेप्यः। अतस्तुल्यशोध्यक्षेपयोर्नाशे कृते शीघ्रोच्चपातयोग एवावशिष्यत इत्युपपन्नम्।

idānīṃ jñaśukrayorviśeṣamāha–

ye cātra pātabhagaṇāḥpaṭhitā jñabhṛgvoḥ te
śīghrakendrabhagaṇairadhikā yataḥ syuḥ |
svalpāḥ sukhārthamuditāścalakendrayuktau
pātau tayoḥ paṭhitacakrabhavau vidheyau ||

...nanu jñaśukrayoḥ śīghroccapātayutiṃ kendraṃ kṛtvā yo vikṣepa ānītaḥ sa śīghroccasthāna eva bhavitumarhati na grahasthāne | yato graho'nyatra vartate | ata idamanupapannamiva pratibhāti |

tathā ca brahmasiddhāntabhāṣye 'jñaśukrayoḥ śīghroccasthāne yāvān vikṣepastāvāneva yatra tatrasthasyāpi grahasya bhavati | atropalabdhireva vāsanā nānyatkāraṇaṃ vaktuṃ śakyata' iti caturvedenāpyadhyavasāyo'tra kṛtaḥ |

satyam | atrocyate | ye'tra jñaśukrayoḥ pātabhagaṇāḥ paṭhitāste śīghrakendrabhagaṇairyutāḥ santastadbhagaṇā bhavanti | tathā ca mādhavīye siddhāntacūḍāmaṇau paṭhitāḥ | ato'lpabhagaṇabhavaḥ pātaḥ svaśīghrakendreṇa yutaḥ kāryaḥ | śīghroccādgrahe śodhite śīghrakendram | tasmin sapāte kṣepakendrakaraṇārthaṃ grahaḥ kṣepyaḥ | atastulyaśodhyakṣepayornāśe kṛte śīghroccapātayoga evāvaśiṣyata ityupapannam |

Now he discusses the peculiarity of Mercury and Venus–

The revolution numbers of the nodes of Mercury and Venus are [actually] more than the ones that have been tabulated here by [an amount equal to] the revolution numbers of the *śīghra*-anomaly. Since these [tabulated values] have been stated to be small for convenience, the tabulated revolution numbers of the nodes of these

> (planets, Mercury and Venus) should be augmented by the revolution numbers of the *śīghra*-anomaly.
>
> ...Is it not true that the latitude, which is obtained considering the sum of the *śīghrocca* and the node as the latitudinal anomaly, can only be the latitude at the location of the *śīghrocca,* and not at the location of the planet, because, the planet is somewhere else. So, this procedure seems clearly unjustified.
>
> As has been stated in the commentary on *Brāhmasphuṭasiddhānta*: 'In the case of Mercury and Venus, whatever is the latitude at the location of *śīghrocca,* the same will be the latitude of the planet, irrespective of whatever be the location of the latter. Here, the only justification is that this is in accordance with observations; and it is not possible to state any other reason. Such was the conclusion reached by even *Caturveda* (Pṛthūdakasvāmin) in this issue.
>
> True. In this matter the following is stated [by way of explanation]: The revolution numbers of the nodes of Mercury and Venus, which have been tabulated here, when augmented by the revolution numbers of the *śīghra*-anomaly will become the [true] revolution numbers of them (nodes). That is how they have been tabulated in the *Siddhāntacūḍāmaṇi* of Mādhava. Thus, the [longitude of the] node computed from the small number of revolutions [as tabulated] should be augmented by the *śīghra*-anomaly of the planet. The *śīghra*-anomaly is obtained by subtracting the planet (*manda-sphuṭa*) from the *śīghrocca.* Having added that to the node, in order to obtain the latitudinal anomaly, the planet (*manda-sphuṭa*) has to be added. If the equal quantities which have been subtracted and added are cancelled, what will finally remain is the sum of *śīghrocca* and the node, and thus (the rule for computing the latitude of Mercury and Venus has been) demonstrated.

As noted earlier, after citing Caturveda Pṛthūdakasvāmin that there is no way of justifying the use of *śīghroccas* in the computation of the latitudes of Mercury and Venus except by the fact that the results tally with observations, Bhāskara tries to provide a rather lame explanation: That the tabulated revolution numbers of the nodes of the interior planets are wrong, and the correction for it exactly compensates the error involved in using the *śīghroccas* in the computation of latitudes.

Now, the standard procedure for computing the latitudes involves the latitudinal anomaly which is given by the sum of the *manda-sphuṭa* (or the *manda*-corrected mean planet) and the node. In the case of Mercury and Venus, the latitudinal anomaly is specially prescribed to be the sum of their *śīghrocca* and the node. Now, according to Bhāskara, the tabulated revolution numbers of the nodes of these planets have to be augmented by the revolution numbers of their *śīghra*-anomalies to get the correct nodes. Hence, Bhāskara concludes that:

The latitudinal anomaly for Mercury and Venus (as prescribed)

$$= \textit{śīghrocca} + \text{node (as tabulated)}$$
$$= \textit{śīghrocca} + \text{true node} - \textit{śīghra}\text{-anomaly}$$
$$= \textit{śīghrocca} + \text{true node} - (\textit{śīghrocca} - \textit{manda-sphuṭa})$$
$$= \textit{manda-sphuṭa} + \text{true node},$$

which is the same as what is also prescribed in the case of the exterior planets. In this way, Bhāskara tries to trivially resolve the paradox of the prevalence of two different rules for computing the latitudes of planets.

5.1 Nīlakaṇṭha's solution of the problem

While the above explanation offered by Bhāskara may sound hardly convincing, that should not detract us from appreciating the fact that he did come up a very clear formulation of the problem. In fact Bhāskara's incisive observation that the latitude which is obtained by employing the *śīghrocca,* 'can only be the latitude at the location of the *śīghrocca,* and not at the location of the planet', seems to have clearly led Nīlakaṇṭha Somayājī, a few centuries later, to come up with the truly revolutionary solution, where the revolution numbers of the nodes of Mercury and Venus were not tampered with, but instead it was their *śīghroccas* which were identified with the mean planets themselves. We can clearly see the influence of Bhāskara in the way Nīlakaṇṭha formulated the problem, while explaining the rationale of the revised planetary model in his *Āryabhaṭīyabhāṣya* [AB1957, pp. 8–9]:

> शीघ्रवशाच्च विक्षेप उक्तः। कथमेतद्युज्यते। ननु स्वबिम्बस्य विक्षेपः स्वभ्रमणवशादेव भवितुमर्हति। न पुनरन्यभ्रमणवशादिति। सत्यम्। न पुनरन्यस्य भ्रमणवशादन्यस्य विक्षेप उपपद्यते। तस्मात् बुधोऽष्टाशीत्यैव दिनैः स्वभ्रमणवृत्तं पूरयति।...एतच्च नोपपद्यते, यत एकेनैव संवत्सरेण तत्परिभ्रमणमुपलभ्यते, नैवाष्टाशीत्या दिनैः। सत्यं, भगोलपरिभ्रमणं तस्याप्येकेनैवाब्देन।...
>
> एतदुक्तं भवति – तयोर्भ्रमणवृत्तेन न भूः कबलीक्रियते। ततो बहिरेव सदा भूः। भगोलैकपार्श्वे एव तद्वृत्तस्य परिसमाप्तत्वात् तद्भगणेन न द्वादशराशिषु चारः स्यात्। तयोरपि वस्तुतः अदित्यमध्यम एव शीघ्रोच्चम्। शीघ्रोच्चभगणत्वेन पठिताः एव स्वभगणाः। तथाप्यादित्यभ्रमणवशादेव द्वादशराशिषु चारः स्यात्।
>
> *śīghravaśācca vikṣepa uktaḥ | kathametadyujyate | nanu svabimbasya vikṣepaḥ svabhramaṇavaśādeva bhavitumarhati | na punaranyabhramaṇavaśāditi | satyam na punaranyasya bhramaṇavaśādanyasya vikṣepa upapadyate | tasmāt budho'ṣṭāśītyaiva dinaiḥ svabhramaṇavṛttaṃ pūrayati |...etacca nopapadyate, yata ekenaiva saṃvatsareṇa tatparibhramaṇamupalabhyate, naivāṣṭāśītyā dinaiḥ | satyaṃ, bhagolaparibhramaṇaṃ tasyāpyekenaivābdena |...*

etaduktaṃ bhavati– tayorbhramaṇavṛttena na bhūḥ kabalīkriyate | tato bahireva sadā bhūḥ | bhagolaikapārśva eva tadvṛttasya parisamāptatvāt tadbhagaṇena na dvādaśarāśiṣu cāraḥ syāt | tayorapi vastutaḥ adityamadhyama eva śīghroccam | śīghrocca bhagaṇatvena paṭhitāḥ eva svabhagaṇāḥ | tathāpyādityabhramaṇavaśādeva dvādaśarāśiṣu cāraḥ syāt |

The latitudinal deflection is said to be due to that of the *śīghrocca*. How is this appropriate? It may be asked: Isn't the latitudinal motion of a body dependent on the motion of that body only? And, not because of the motion of something else? True, the latitudinal motion of one body cannot be obtained as being due to the motion of another. Hence [we should conclude that] Mercury goes around its own orbit in 88 days...However, this also is not appropriate, since we see it (Mercury) going around [the Earth] in one year and not in 88 days. True, the period in which Mercury completes one full revolution around the sphere of stars is one year only [like the Sun]...

This is what is stated (by us in explanation): Their (Mercury and Venus) orbits do not swallow up the earth. The Earth is always outside their orbit. Since their orbit is always confined to one side of the [geocentric] celestial sphere, in completing one revolution they do not go around the twelve signs (*rāśis*). Actually, for them (Mercury and Venus) also the mean Sun is the *śīghrocca* [as is the case for Mars, Jupiter and Saturn]. It is only their own revolutions which have been tabulated as the revolution numbers of the *śīghroccas*. Even so, it is only due to the revolution of the Sun [around the Earth], that they (Mercury and Venus) complete their movement around the twelve signs [and complete their revolution of the Earth].

6 Concluding Remarks

The works of Bhāskarācārya *viz.*, *Līlāvatī, Bījagaṇita* and *Siddhāntaśiromaṇi,* very deservedly acquired the status of canonical textbooks. They were appreciated for their clarity, poetic elegance and beauty, and also for their comprehensiveness in covering the entire gamut of mathematics and astronomy as developed in the Indian tradition in the *siddhāntic* period. Hundreds of manuscripts of these works are still available in various collections. Scores of commentaries were written and a number of translations into various languages were also carried out in the pre-modern era.

The *Vāsanābhāṣyas* of Bhāskara also became the exemplars for the presentation of detailed explanations and justifications. The vast commentarial literature, which emerged in the post-Bhāskara period, is clearly inspired by the *Vāsanābhāṣya* on the *Siddhāntaśiromaṇi.* As we have tried to explain in this article, the *Vāsanābhāṣyas* of Bhāskara also expound upon many of the basic issues which are crucial for understanding the methodology of mathematics and astronomy in the Indian tradition. This tradition did continue also in the post-Bhāskara period, as can be seen in some of the seminal works

of the Kerala School, such as the *Āryabhaṭīyabhāṣya* of Nīlakaṇṭha Somayājī and the *Gaṇitayuktibhāṣā* of Jyeṣṭhadeva.

Our discussion also indicates how the works of Bhāskara did have a significant impact on later developments in mathematics and astronomy, an impact which was indeed pan-Indian in nature. Of course, it is well known that the influence of Bhāskara is clearly visible in most of the subsequent work on mathematics and astronomy carried out in north India. This can be seen as much in the mathematical works of Nārāyaṇa Paṇḍita or the commentaries of Gaṇeśa Daivajña and Kṛṣṇa Daivajña, as in the large corpus of works on mathematical astronomy created by several families of astronomers hailing from places such as Dadhigrāma, Nandigrāma, Golagrāma etc., in western India, and their descendents and disciples many of whom later settled in Vāraṇāsi. It is equally important to note that much of the work of the Kerala School of astronomy, which clearly drew inspiration from the tradition of Āryabhaṭa, was also built on the edifice created by Bhāskara by means of his canonical text books on mathematics and astronomy. We have tried to highlight in this paper how some of the important achievements of the Kerala astronomers such as Mādhava's exact analytic expression for the iterated-*manda-karṇa,* Parameśvara's geometrical model of planetary motion, and even Nīlakaṇṭha's revised planetary model can be traced to the clear and penetrating analysis of all these issues that we find in the *Vāsanābhāṣya* of Bhāskarācārya on *Siddhāntaśiromaṇi.*

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The production of Sines: Bhāskarācārya's *Jyotpatti*

Clemency Montelle*

1 Introduction

Jyotpatti, literally 'Production of Sines', is the name given to one of the chapters of Bhāskara's *Siddhāntaśiromaṇi* (1150 CE). The chapter is twenty-five verses long and it deals exclusively with trigonometry.[1] In it, Bhāskara sets out various conventions and relations for producing Sine values for various arcs. While some of the formulae he presents had been long known and used (notably in verses 1–10), Bhāskara also produces a group of previously unattested rules culminating in the notable Sine and Cosine addition and subtraction formula. As with the other chapters of the *Siddhāntaśiromaṇi*, Bhāskara writes a commentary on this chapter. This includes descriptions on how to compute various Sine values, which formulae to use, and the advantages of some formulae over others.

The contents of this chapter raise many interesting questions about the role and nature of trigonometry by Bhāskara's time in the astral sciences in Sanskrit sources. The most notable is the fact that here trigonometry is

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[1] This chapter has been translated into English by [SL1861, pp. 263–268] and discussed in [Plo2009, pp. 204–205], [Van2009, pp. 105–107], [Gup1974], [Gup1976], and [Yan1977] Other studies on the *Jyotpatti* include the Marīci commentary on this work by [Gup1980]. For broader studies on the history of trigonometry in this tradition, see for instance, [KhKh1970, Appendix VI, pp. 176–188] and [DA1983, pp. 39–108]. This study has used the edition in [SiŚi1989, pp. 281–286] .

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treated as an independent topic. Prior to this, precomputed trigonometric quantities were always found in the midst of astronomical content, commonly in chapters dedicated to the determination of true planetary positions and velocities. Indeed, it is true that some other earlier works in the Sanskrit astral sciences did include in separate places a justification of Sine values that had been given earlier in the text. Brahmagupta, for instance, includes four verses in chapter twenty-one of the *Brahmasphuṭasiddhānta* (XXI, 19–23) that offer rules to produce the versified Sines given in chapter two (II, 2–5). However, the *Jyotpatti* goes beyond this. While Bhāskara provides the rules for Sines that appear earlier in the *Siddhāntaśiromaṇi*, many rules he includes can be used to generate Sines that are not found anywhere else in his corpus. Thus, the significance of the *Jyotpatti* being its own separate chapter accentuates the gradual expansion of the purview of this topic at this time.

In order to appreciate certain features of this development, I considered the collection of mathematical rules presented in the *Jyotpatti* and the ways in which the rules complement and contrast one another. Notably, from a strictly mathematical viewpoint, many of the rules Bhāskara presents in this chapter are equivalent or alternatives for the same expression. Why, then, does he present so many rules which overlap? What does this reveal about the purpose of this chapter? Is this a display of mathematical virtuosity above and beyond a practical guide to compute Sine tables? Exploring this issue allows us a glimpse into the scope of this chapter, including who the text was directed towards, the ways in which it was intended to be used, as well as shedding some light on the complex evolving role of trigonometry as a subject in the astral sciences.

2 The *Jyotpatti*

Jyotpatti is a compound of the Sanskrit words *jyā*, here Sine, and *utpatti*, a feminine noun meaning production, origin, arising, birth. *Utpatti* in the context of the astral sciences is a technical term used to indicate that the subject matter concerns the 'production' of the mathematical items in question.[2] Bhāskara includes trigonometric material elsewhere in the *Siddhāntaśiromaṇi*. Earlier in this work, he presents an enumeration of versified Sines (and Versines) in the chapter on true motions (*spaṣṭādhikāra* (2-9)) in which 24 Sines are given with

[2] For instance, the twelfth century commentator on Brahmagupta, Āmarāja, invokes the term *utpatti* to describe those sections of his commentary which carry out numerical procedures or the algorithms relating to the resulting numerical data. In particular, he uses this term to introduce the section where he describes the production of Brahmagupta's Sines [KhKh1925, pp. 98–100].

a Radius of 3438. In another work, the *Karaṇakutūhala*, Bhāskara provides a different versified table of Sines where R=120 with ten values (2, 6-8), again in the chapter which deals with the true positions of the planets.

Table 1: The trigonometric rules in the order they appear in the *Jyotpatti*.

Verse No.	Content of the verses	Eq. no.
1	Invocation	
2	Constructing the diagram	
3	Diagrammatic definition of the Sine	
4	$\text{Cos}\theta = \sqrt{R^2 - \text{Sin}^2\theta}$	(i)
5	$\text{Vers}\,\theta = R - \text{Cos}\theta$	(ii)
	$\text{Vers}\,\overline{\theta} = R - \text{Sin}\theta$	(iia)
6	$\text{Sin}30 = \frac{R}{2} = \text{Cos}60$	(iii)
	$\text{Sin}45 = \sqrt{\frac{R^2}{2}}$	(iv)
7	$\text{Sin}36 = \sqrt{\frac{5R^2 - \sqrt{5R^4}}{8}}$	(v)
8	$\text{Sin}36 = \text{Cos}54 \approx \frac{5878}{10000} \cdot R$	(vi)
9	$\text{Sin}18 = \frac{\sqrt{5R^2} - R}{4}$	(vii)
10	$\text{Sin}\left(\frac{\theta}{2}\right) = \frac{\sqrt{\text{Sin}^2\theta + \text{Vers}^2\,\theta}}{2} = \sqrt{\frac{R\,\text{Vers}\,\theta}{2}}$	(viii)
12	$\text{Sin}\left(\frac{90 \pm \theta}{2}\right) = \sqrt{\frac{R^2 \pm R\,\text{Sin}\theta}{2}}$	(ix)
13	$\text{Sin}\left(\frac{\theta - \phi}{2}\right) = \frac{\sqrt{(\text{Sin}\theta - \text{Sin}\phi)^2 + (\text{Cos}\phi - \text{Cos}\theta)^2}}{2}$	(x)
14	$\text{Sin}\left(\frac{\theta - \overline{\theta}}{2}\right) = \sqrt{\frac{(\text{Sin}\theta - \text{Cos}\theta)^2}{2}}$	(xi)
15	$\text{Sin}(\theta - \overline{\theta}) = R - \frac{2\,\text{Sin}^2\theta}{R}$	(xii)
16-18a	$\text{Sin}(\theta \pm 1) = \left(1 - \frac{1}{6569}\right)\text{Sin}\theta \pm \frac{10}{573}\,\text{Cos}\theta$	(xiii)
	for 90 Sines where $R = 3438$ and Sin1=60	
18b-20	$\text{Sin}(\theta \pm 3;45) = \left(1 - \frac{1}{467}\right)\text{Sin}\theta \pm \frac{100}{1529}\,\text{Cos}\theta$	(xiv)
	for 24 Sines where $R = 3438$ and Sin3;45=225-$\frac{1}{7}$	
21-24	$\text{Sin}(\theta \pm \phi) = \frac{\text{Sin}\theta\,\text{Cos}\phi \pm \text{Sin}\phi\,\text{Cos}\theta}{R}$	(xv)
25	Closing verse	

The trigonometric rules as they appear in the *Jyotpatti* go above and beyond providing ways to generate these versified Sines Bhāskara provides. They can be summarized (using modern notation) in table 1. What is immediately noticeable about this group of rules is the variety. Not only are there rules to find the Sines of various fixed arcs (30, 45, 36, and so on) for an arbitrary Radius, but also rules for half-arcs, rules involving the arc and its complement, rules for the sum or difference of an arbitrary arc and a fixed amount, and the Sine and Cosine sum-and-difference formula.

Bhāskara states at the outset that this chapter is for those who wish to compute a Sine table with any number of data points and whatever Radius:

> इष्टाङ्गुलव्यासदलेन वृत्तं कार्यं दिगङ्कं भलवाङ्कितञ्च ।
> ज्यासंख्ययाप्ता नवतेर्लवा ये तदाद्यजीवाधनुरेतदेव॥ २ ॥
>
> *iṣṭāṅgulavyāsadalena vṛttaṃ kāryaṃ digaṅkaṃ bhalavāṅkitañca* |
> *jyāsaṃkhyayāptā navater lavā ye tadādyajīvādhanuretadeva* ||2||
>
> A circle with radius [equal to] the desired digits [*aṅgulas*] and with directions and 360 [equal] divisions marked on it is to be drawn. Whatever [results] from ninety degrees divided by the number of Sines [one wishes to generate], this is the arc of the first Sine.
>
> [SiŚi1989, p. 281, *Jyotpatti* v. 2]

Different graduations in arc require different formula and the group of rules one requires to compute each arc will depend on the number of data points one selects. This is confirmed by Bhāskara's commentary, in which he gives instructions for the computation of several Sine tables each with a varying number of entries. For one who wishes to construct a Sine table with 6 entries, one requires the rules (i-iv), for the canonical trigonometrical table with 24 entries, one requires rules (i-iv) and (viii). In order to produce 30 Sines, one requires (i-x), and so on.

At the beginning of the treatise, Bhāskara instructs the reader to construct a circle and to depict the Sines with reference to this circle. He is explicit that the Sines are half of a piece of string with length equal to the cord of the appropriate arcs, and directly states that all the Sines can be found this way.

> ... चापे तु दत्वोभयत्रो दिगङ्कात्।
> ज्ञेयं तदग्रद्वयबद्धरज्ञोरर्धं ज्यकार्धं निखिलानि चैवम्॥ ३ ॥
>
> ... *cāpe tu datvobhayatro digaṅkāt* |
> *jñeyaṃ tadagradvayabaddharajjor ardhaṃ jyakārdhaṃ nikhilāni caivam* || 3 ||
>
> ...Having produced [these arcs] on the arc on either side of a compass point, the Sine [*jyakārdham*] is understood as half of the cord attached to the two tips of this [arc]. All [Sines are to be established] in this way.
>
> [SiŚi1989, p. 281, *Jyotpatti* v. 3]

Whether or not this is intended to reflect an actual way of determining Sines, via measures of the lengths of pieces of strings, is arguable. Perhaps it is simply a reference to the diagram with which the mathematical rules can be understood. However this passage appears to be explicitly in contrast with the 'arithmetical' techniques the immediately follow which may argue for the former interpretation. Indeed, in the very next verse, Bhāskara states that the Sines can be found in another way (*anyathā*), namely using arithmetic (*gaṇita*). Furthermore, this procedure will produce more accurate results (*parisphuṭa*):

> अथान्यथा वा गणितेन वच्मि ज्यार्धानि तान्येव परिस्फुटानि ...॥ ४ ॥
>
> *athānyathā vā gaṇitena vacmi jyārdhāni tānyeva parisphuṭāni...* || 4 ||
>
> Now, in another way, I declare by means of arithmetic these fully-accurate Sines.
>
> [SiŚi1989, p. 281, *Jyotpatti* v. 4 (first half)]

The three details—'another way', 'by arithmetic' and 'fully accurate'—may be intended to stand in contrast to a less accurate geometric method.

To appreciate how Bhāskara may imagine his text to be used, we consider an excerpt from his commentary. The first example he gives is how to compute a Sine table with 24 entries. He explains:

> ...Whenever there are twenty-four Sines, in that case, half the Radius is the 8th Sine. The Cosine of that is the 16th. The Sine of 45 is the 12th. Now, from the 8th by the rule of the degrees of half of that, the 4th. The Cosine of that is the 20th. In the same way from the 4th, the 2nd and the 22nd. From the 2nd, the 1st and the 23rd. From the 20th, the 10th and the 14th. From the 10th, the 5th and the 19th. From the 22nd, the 11th and the 13th. From the 14th, the 7th and the 17th. Now from the 12th, the 6th and the 18th. From the 6th the 3rd and the 21st. From the 18th, the 9th and the 15th. The Radius is the 24th. In this way indeed the accomplishment of the different Sines by the ancients is stated.
>
> [SiŚi1989, (Bhāskara's Com. p. 284, lines 2-9)]

To compute these you need rules (i) to (viii). 'Half the Radius is the 8th Sine' refers to rule (iii); 'The Sine of 45 is the 12th' refers to rule (iv); the Cosines are to be found, presumably, using (i); and the phrase 'by the rule of the degrees of half of that,' he is alluding to rule (viii). Interestingly enough, one does not require the Sine of 36 (v)/(vi) nor 18 (vii) to compute this table, yet they are interspersed in this group.

Another point of interest is the way in which Bhāskara refers to the Sines. Rather than referring to them by means of the measure of their arc (i.e., the Sine of $3\frac{3}{4}$, and so on) he references them using ordinal numbers (the only exception is the Sine of 45 degrees which he states is the 12th Sine). This is not Bhāskara's innovation; Brahmagupta refers to Sines in exactly the same

way (*Brahmasputasiddhānta* XXI 19–23). This means to refer to the Sines emphasizes their relatedness as a collection of ordered data. It also has the consequence that applying the half angle formula and various symmetry considerations is obvious. For instance, any even numbered ordinal can produce new Sines by the half-angle formula. However, one must be sure of the context: the first Sine when one is computing 24 Sines, is **not** the same as the first Sine when computing, say, 10 Sines.

Next, Bhāskara in his commentary poses a new problem: suppose one wants '30 Sines'.

> यत्र किल त्रिंशज्ज्यार्द्धानि तत्र त्रिज्यार्धं दशमम् ।
>
> *yatra kila triṃśajjyārddhāni tatra trijyārdhaṃ daśamam* |
>
> Whenever there are thirty Sines, there, the tenth is half of the Radius.
>
> [SiŚi1989, (Bhāskara's Com. p. 284, line 23)]

He runs through the same procedure, and shows how the traditional methods will give you the Sines for (not in order of how they are produced) 3rd, 5th, 6th, 9th, 10th, 12th, 15th, 18th, 20th, 21st, 24th, 25th, 27th.

Notably, the method for computing the the 12th (i.e., the Sine of 36) is given as per (v), but the method for computing the 6th (i.e., the Sine of 18) is used from applying the half-angle formula to the 12th and not (vii). Once he has run through these, Bhāskara notes:

> एतान्येवानेन प्रकारेण सिद्ध्यन्ति नान्यानि
>
> *etānyevānena prakāreṇa siddhyanti nānyāni* |
>
> Only these [Sines] are obtained by means of this method, others are not.
>
> [SiŚi1989, (Bhāskara's Com. p. 284, line 29)]

For the remaining Sines, he calls upon the rule (x):

$$\text{Sin}\left(\frac{\theta - \phi}{2}\right) = \frac{\sqrt{(\text{ Sin}\theta - \text{ Sin}\phi)^2 + (\text{ Cos}\phi - \text{ Cos}\theta)^2}}{2}$$

using the 5th and the 9th to find the 2nd, and from the 2nd the 28th. Then, from the traditional half-angle formula, one can find the 14th and the 1st, and he notes "the other 14 are established in the same way".

In this way, the exposition in the commentary reveals the progressive addition of rules to find Sine tables with an increasing number of data points. His last example works through a Sine table with 90 data points and makes extensive use of the Sine-Cosine addition and subtraction formula (xv), the very last rule on his list.

3 Why so many different formulae?

Given the fact that the *Jyotpatti* is directed towards practitioners who want to construct their own Sine table of whatever size and with whatever Radius, the variety of rules to compute the Sines of many different arcs is appropriate. However, a careful perusal of the collection of rules reveals that for some arcs, there are often several formulae that can be invoked. This can be seen trivially in the alternatives given for, say, the Sine of 36:

$$\mathrm{Sin}36 = \sqrt{\frac{5R^2 - \sqrt{5R^4}}{8}} = \mathrm{Cos}54 \approx \frac{5878}{10000} \cdot R$$

where $\frac{5878}{10000}$ is an approximation to $\sqrt{\frac{5-\sqrt{5}}{8}}$. Less trivially, once one has the Sine of any value in the quadrant (when $R = 3438$), and then the rule for $\mathrm{Sin}(\theta \pm 1)$ (rule xiii) will easily provide the rest if single degrees (or multiples thereof) are required. Or, for instance, once one has the Sine of any multiple of 3;45 degrees (which is the division if you require a Sine table with the canonical 24 values), then all one requires is rule (xiv), $\mathrm{Sin}(\theta \pm 3;45)$, to generate the others. In addition, there are direct equivalents of some rules by savvy substitutions. For instance, rule (ix) can be produced from substituting $\theta = 90$ into rule (x), or rule (xi) from rule (x), and so on. In a related way, rules (xiii) and (xiv) can be produced from rule (xv).

How are we to interpret this collection of rules, therefore? Why does Bhāskara include them all, given that many of them are essentially mathematically equivalent, when a smaller group would suffice for the objective of compiling a Sine table with some number of entries and an arbitrary Radius?

I offer several plausible (and complementary) reasons for this. The first is essentially archival. Bhāskara is summarizing some of the rules his predecessors used (verses 4-10: rules (i) to (viii)). He credits these rules to them, naming them as *pūrvaiḥ (ganakaiḥ)*, literally 'earlier (mathematicians)'. He is not however explicit as to exactly who he means; Varāhamihira, Brahmagupta, and Lalla are no doubt meant to be included in this group.

The remaining rules he claims as his own, and, in addition, he declares them to be superior:

> ...प्रवक्ष्येऽथ विशिष्टमस्मात्
>
> *...pravakṣye 'tha viśiṣṭam asmāt*
>
> Now, I will state (the acquisition method) better than this.
>
> [SiŚi1989, p. 281, *Jyotpatti* v. 11 (excerpt)]

The second way in which to comprehend the integrity of this collection is based on the specific arithmetical operations invoked in the rules. At several points in both the commentary and the verses, Bhāskara makes the several telling claims.[3] At one point in the commentary he notes one feature of a rule is that fact that it avoids using the Versine :

> इदानीं विनाप्युत्क्रमज्ययाभिनवप्रकारेणाह ।
>
> *idānīṃ vināpy utkramajyayābhinavaprakāreṇāha* |
>
> Now, he (i.e., Bhāskara) expounds [the Sines] with a new approach without even [using] the Versine.
>
> [SiŚi1989, (Bhāskara's Com. p. 284, line 10)]

At another point in a verse, he claims (before he articulates rules (xii-xvi)) that the rules that follow avoid taking the square-root:

> ...वक्ष्येऽथ मूलग्रहणं विनापि॥ १४ ॥
>
> *...vakṣye'tha mūlagrahaṇaṃ vināpi* || 14 ||
>
> Now, I will state [the Sines] without even taking a square root.
>
> [SiŚi1989, p. 282, *Jyotpatti* v. 14 (excerpt)]

He reiterates this again in his commentary:

> अथ मूलग्रहणक्रियया विनापि दोःकोटिभागान्तरज्यानयनमाह ।
>
> *atha mūlagrahaṇakriyayā vināpi doḥkoṭibhāgāntarajyānayanam āha* |
>
> Now, he says the computation of the Sine of the difference of the degrees of the arc and the complement without even computing a square root.
>
> [SiŚi1989, (Bhāskara's Com. p. 284, line 21)]

Indeed, the last group of Sine rules ((xii)—(xv)) do not require square roots. In summary, it appears that Bhāskara is offering alternative rules because they replace one arithmetic operation with a different, less computationally complex one. More generally, Bhāskara provides equivalent rules which avoid certain procedures, such as computing a square root or a Versine, probably because these latter operations are computationally more involved. In this way, he demonstrates a consideration of the practical demands of computation; his alternatives appear to reduce the computational burden of his reader, and possibly also to improve the accuracy of the results.[4]

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[3] Bhāskara adopts a notable commentarial technique—that is, he refers to himself in the commentary using the third person, rather than the first person as one might expect. So all references to "he" in the commentary almost always refer to himself. This practice is typical in the Indian tradition.

[4] Offering alternative methods that avoid square roots appear in other contexts as well. See, for instance, sections 6.2 and 6.3 of [GaYu2008, vol. 1, pp. 180–192] in which a

Indeed, another form of implicit equivalence are the two formula Bhāskara gives almost at the end ((xiii) and (xiv)):

$$\mathrm{Sin}(\theta \pm 1) = \big(1 - \frac{1}{6569}\big)\,\mathrm{Sin}\theta \pm \frac{10}{573}\,\mathrm{Cos}\theta,$$
$$\mathrm{Sin}(\theta \pm 3;45) = \big(1 - \frac{1}{467}\big)\,\mathrm{Sin}\theta \pm \frac{100}{1529}\,\mathrm{Cos}\theta.$$

These rules offer a single formula from which to generate a Sine table of 90 or 24 entries, rather than having to invoke a number of rules depending on the arc. Here too Bhāskara is in part motivated by reducing the computational effort on the part of the computer by providing highly accurate approximations to the appropriate expressions. It is likely that these two rules were generated from the more general addition and subtraction formula:

$$\mathrm{Sin}(\theta \pm \phi) = \frac{\mathrm{Sin}\theta\,\mathrm{Cos}\phi \pm\ \mathrm{Sin}\phi\,\mathrm{Cos}\theta}{R}$$

by substituting $\phi = 1$ and 3;45 respectively. Substituting in carefully selected Sines for a known arc can generate these rules. In the former case, one can note that by the above formula:

$$\mathrm{Sin}(\theta \pm 1) =\ \mathrm{Sin}\theta \cdot \frac{\mathrm{Cos}1}{R} \pm\ \mathrm{Cos}\theta \cdot \frac{\mathrm{Sin}1}{R},$$

where

$$\frac{\mathrm{Sin}1}{R} = \frac{60}{3438} = \frac{10}{573} \text{ and } \frac{\mathrm{Cos}1}{R} \approx 1 - \frac{1}{6569}.$$

Similarily,

$$\mathrm{Sin}(\theta \pm 3;45) =\ \mathrm{Sin}\theta \cdot \frac{\mathrm{Cos}3;45}{R} \pm\ \mathrm{Cos}\theta \cdot \frac{\mathrm{Sin}3;45}{R},$$

where

$$\frac{\mathrm{Sin}3;45}{R} = \frac{225 - \frac{1}{7}}{3438} \approx \frac{100}{1529} \text{ and } \frac{\mathrm{Cos}3;45}{R} \approx 1 - \frac{1}{467}.$$

Thus, Bhāskara's exposition of Sine rules embodies several desiderata. Firstly is indeed a display of technical brilliance. But just as important practical considerations appear to have been a priority. These include the provision of a variety of rules so that the reader can construct their own Sine table with the Radius they desire as well as the number of data points. In addition to this, Bhāskara offers a range of mathematically equivalent rules so that the practitioner can avoid tricky arithmetical procedures.

method to compute the circumference of a circle is offered which avoids the use of square roots.

4 Conclusion

Bhāskara's *Jyotpatti* offers a range of formula to compute Sine tables of various sizes and Radii. The term *utpatti* here emphasizes the practical orientation of this text: instructions for the construction of Sines via a diagram and algorithms to compute Sines tailored to certain parameters above and beyond those versified Sines given elsewhere by Bhāskara. Given the unique nature of the *Jyotpatti* it is no surprise that various recensions of the *Siddhāntaśiromaṇi* have included this chapter in different places. Later commentators and scribes appear to have been unsure as to where to place this chapter in this work;[5] sometimes it appears at the end of the *Siddhāntaśiromaṇi*, in some versions it is included early in the work, for instance. This is a significant point and offers insight into the ways in which this text was received and accounted for by later practitioners. Future studies which elaborate on this mobility, once a large selection of the various manuscripts have been consulted, will further enrich our appreciation and understanding of this critical and curious chapter.

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[5] [Pin1981, p. 29] notes it is 'variously placed in various editions'.

An application of the addition and subtraction formula for the Sine in Indian astronomy

Setsuro Ikeyama*

1 Introduction

It is well known that Bhāskara is the first Indian astronomer who mentioned clearly the addition and subtraction formula for the sine and the cosine. Bhāskara himself did not give any rationale but several astronomers who followed him explained and utilized the formula. We recently found an astronomical application of the formula (hereafter 'application formula') by the famous Kerala astronomer-mathematician Mādhava and its rationale by Nīlakaṇṭha Somasutvan in Nīlakaṇṭha's works composed in the early sixteenth century. The application formula is for adding and subtracting two sines which are not in the same plane and are utilized for computing the declination of a planet when it deflects from the ecliptic.

In this paper, we will introduce and discuss the application formula given by Mādhava and its rationale given by Nīlakaṇṭha. We quote the Sanskrit texts from the published editions; we apply a few corrections shown in square brackets and we silently amend some small and obvious mistakes such as erroneous *sandhi*.

2 Bhāskara's addition-subtraction formula for the Sine

Bhāskara clearly stated the addition and subtraction formula for the sine in three verses of the section called 'Generation of Sines (*jyotpatti*)' in his *Siddhāntaśiromaṇi* [SiŚi1981, pp. 527–528].

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चापयोरिष्टयोर्दोर्ज्ये मिथः कोटिज्यकाहते।
त्रिज्याभक्ते तयोरैक्यं स्याच्चापैक्यस्य दोर्ज्यका॥ २१ ॥
चापान्तरस्य जीवा स्यात् तयोरन्तरसंमिता।
अन्यज्यासाधने सम्यग् इयं ज्याभावनोदिता॥ २२ ॥
समासभावना चैका तथान्यान्तरभावना। २३ ।

cāpayor iṣṭayor dorjye mithaḥ koṭijyakāhate |
trijyābhakte tayor aikyaṃ syāc cāpaikyasya dorjyakā || 21 ||
cāpāntarasya jīvā syāt tayor antarasaṃmitā |
anyajyāsādhane samyag iyaṃ jyābhāvanoditā || 22 ||
samāsabhāvanā caikā tathānyāntarabhāvanā | 23ab|

> The Rsines (*dorjyā*) of the two wished arcs multiplied mutually by their side Rcosines (*koṭijyakā*) and divided by the Rsine of three [zodiacal signs] (i.e., by the radius). The sum of those two must be the Rsine of the sum of the two arcs.
>
> The Rsine of the difference of the two arcs would be measured by the difference of those two. This generation method (*bhāvanā*) of Rsines has been correctly told for the computation of other Rsines.
>
> One is the generation method of the sum and the other is the generation method of the difference.

At first, we will list Rsines and Rcosines and corresponding lines in Figure 3:

$$\begin{aligned} \mathrm{Sin} ZV &= BV & \mathrm{Cos} ZV &= OB \\ \mathrm{Sin} VP &= PU & \mathrm{Cos} VP &= OU \\ \mathrm{Sin}(ZV + VP) &= B'P & OV = OP = OD &= R \end{aligned}$$

where $\mathrm{Sin} ZV$ and $\mathrm{Cos} ZV$ mean the Rsine and Rcosine of arc ZV respectively. Let N be the foot of the perpendicular from U to $B'P$.

In Figure 3, where $ZV > VP$, Bhāskara's formula is:

$$\mathrm{Sin}(ZV + VP) = \frac{\mathrm{Sin} ZV \cdot \mathrm{Cos} VP}{R} + \frac{\mathrm{Sin} VP \cdot \mathrm{Cos} ZV}{R}. \tag{1}$$

Bhāskara also mentions the subtraction formula, which is valid for the case when P comes between Z and V.

$$\mathrm{Sin}(ZV - VP) = \frac{\mathrm{Sin} ZV \cdot \mathrm{Cos} VP}{R} - \frac{\mathrm{Sin} VP \cdot \mathrm{Cos} ZV}{R}. \tag{2}$$

3 Mādhava's expression of the addition-subtraction formula and its demonstration by Nīlakaṇṭha

Several Indian mathematicians/astronomers refer to and rationalize the addition-subtraction formula introduced by Bhāskara. Here we take up the discussion made among the "Mādhava School".

Text and Translation

The formula is versified by Mādhava (fl. ca. 1380/1420), and is quoted by Nīlakaṇṭha (b. 1444, d. 1542) in his *Tantrasaṅgraha* [TaSa1977, ch. 2.16, p. 125]:

> जीवे परस्परनिजेतरमौर्विकाभ्यां
> अभ्यस्य विस्तृतिदलेन विभाज्यमाने ।
> अन्योन्ययोगविरहानुगुणे भवेतां
> यद्वा स्वलम्बकृतिभेदपदीकृते द्वे ॥
>
> *jīve parasparanijetaramaurvikābhyāṃ*
> *abhyasya vistṛtidalena vibhājyamāne |*
> *anyonyayogavirahānuguṇe bhavetāṃ*
> *yadvā svalambakṛtibhedapadīkṛte dve ||*

And also in his *bhāṣya* (NAB) on the *Āryabhaṭīya Gaṇitapāda* [AB1930, pt. 1, p. 58, lines 21–26]:

> तद्विषयं वसन्ततिलकं संङ्गमग्रामजमाधवनिर्मितं पद्यं च श्रुतम्। यथा –
>
> जीवे परस्परनिजेतरमौर्विकाभ्यां
> अभ्यस्य विस्तृतिगुणेन विभज्यमाने ।
> अन्योन्ययोगविरहानुगुणे भवेतां
> यद्वा स्वलम्बकृतिभेदपदीकृते द्वे ॥
>
> *tadviṣayaṃ vasantatilakaṃ saṅgamagrāmajamādhavanirmitaṃ padyaṃ ca śrutam | yathā –*
>
> *jīve parasparanijetaramaurvikābhyāṃ*
> *abhyasya vistṛtiguṇena*[1] *vibhajyamāne |*
> *anyonyayogavirahānuguṇe bhavetāṃ*
> *yadvā svalambakṛtibhedapadīkṛte dve ||*
>
> A verse in the *vasantatilakā* meter with regard to it (i.e., the sum of the Rsines), which was made by Mādhava born in Saṅgamagrāma, is also heard. That is:

[1] Read *dalena* for *guṇena* as in [TaSa1977, ch. 2.16] above and in the quotation by Śaṅkara in his *Kriyākramakarī* [Līlā1975].

> Two Rsines mutually multiplied by the other Rsines (i.e., Rcosines) and divided by radius will be fit for the mutual addition and subtraction. Or, the two which are the square roots of the differences of the squares of themselves and of the perpendiculars (*lamba*) (will also be fit for the addition and subtraction).

This verse included two different expressions of the addition formula. The first three lines mention the one we discuss in this paper.[2] The formula is as follows. When $ZV > VP$,

$$\mathrm{Sin}(ZV \pm VP) = \frac{\mathrm{Sin}ZV \cdot \ \mathrm{Cos}VP}{R} \pm \frac{\mathrm{Sin}VP \cdot \ \mathrm{Cos}ZV}{R},$$

the same as Bhāskara's one (equations (1) and (2)).

Demonstration

Nīlakaṇṭha demonstrates Mādhava's formula in NAB Part I [AB1930, p. 59, line 4ff]. We explain its derivation using Figure 3. For a detailed discussion see [Gup1974, pp. 169–171].

In Figure 3, from the three similar triangles, OBV, $OB''U$, PNU, we obtain the following two proportions:

$$OV : BV :: OU : B''U$$
$$OV : OB :: PU : PN.$$

And because $B''B'NU$ is a rectangle, it is obvious that $B''U = B'N$.

Using these relations, we have

$$B'N = \frac{BV \cdot OU}{OV},$$
$$PN = \frac{PU \cdot OB}{OV}.$$

Therefore,

$$\begin{aligned} B'P &= B'N + PN \\ &= \frac{BV \cdot OU}{OV} + \frac{PU \cdot OB}{OV}, \end{aligned} \tag{3}$$

[2] Another formula mentioned in the last line is discussed by Nīlakaṇṭha in [AB1930]. See [Gup1974, pp. 164–177] and [Hay1997a, pp. 199–207].

that is,

$$\mathrm{Sin}(ZV + VP) = \frac{\mathrm{Sin}ZV \cdot \mathrm{Cos}VP}{R} + \frac{\mathrm{Sin}VP \cdot \mathrm{Cos}ZV}{R}.$$

4 Application of the formula by Mādhava

Nīlakaṇṭha discusses Mādhava's application of the addition-subtraction formula in his *bhāṣya* on the *Āryabhaṭīya Golapāda* [AB1957], We select the Sanskrit passages that show his line of argument, translate them into English, and give our mathematical comments with the help of Figures 1 and 2.

Introduction of the application formula

First Nīlakaṇṭha cites Mādhava's verses which prescribe the application formula [AB1957, p. 108, lines 8–15]:

> अपक्रमचापविक्षेपचापयोर्योगो वियोगो वा न स्फुटापक्रमचापं स्यात्। अत एव गोलविदा माधवेन विक्षेपवतां स्फुटापक्रमानयने गणितविशेषः प्रदर्शितः।
>
> > परमापक्रमकोट्या विक्षेपज्यां निहत्य तत्कोट्या।
> > इष्टक्रान्तिं चोभे त्रिज्याप्ते योगविरहयोग्ये स्तः॥
> > सदिशोः संयुतिरनयोः वियुतिर्विदिशोरपक्रमः स्पष्टः।
> > स्पष्टापक्रमकोटिर्द्युज्या विक्षेपमण्डले वसताम्॥ इति ॥
>
> *apakramacāpavikṣepacāpayor yogo viyogo vā na sphuṭāpakramacāpaṃ syāt | ata eva golavidā mādhavena vikṣepavatāṃ sphuṭāpakramānayane gaṇitaviśeṣaḥ pradarśitaḥ |*
>
> > *paramāpakramakoṭyā vikṣepajyāṃ nihatya tatkoṭyā |*
> > *iṣṭakrāntiṃ cobhe trijyāpte yogavirahayogye staḥ ||*
> > *sadiśoḥ saṃyutir anayoḥ viyutir vidiśor apakramaḥ spaṣṭaḥ |*
> > *spaṣṭāpakramakoṭir dyujyā vikṣepamaṇḍale vasatām || iti ||*
>
> The sum or difference of the arc of declination (*apakrama*) and the arc of latitude (*vikṣepa*) does not give the true declination. Therefore, Mādhava, who knows spheres (*gola*), explained a special computation for deriving the true declinations (of the planets) which have latitudes:
>
> > Multiply the Rsine of the latitude (PU) by the side of the maximum declination and [the Rsine] of the given declination by its side (OU), and divide those two by the radius (R). [The results] will be fit for addition or subtraction.

> If the directions of these two are same, sum is [the Rsine of] the true declination ($C'P$), when different, the difference.
> The side of the true declination is the radius of diurnal circle of those on the latitude circle.[3]

For the purpose of simplifying the argument, Nīlakaṇṭha assumes the situation where observers are at the terrestrial equator and the foot of the perpendicular from a planet P to the ecliptic plane lines in the plane containing the nonagesimal V (the middle point of the half of the ecliptic which is above the horizon at a given moment). Figure 1 shows that particular situation. Here points Z, V, P, and D lie on the circle called *dṛkkṣepamaṇḍala* in Sanskrit. The nonagesimal V and arc ZV are called *dṛkkṣepalagna* and *dṛkkṣepa* respectively. The *dṛkkṣepamaṇḍala* and the ecliptic are at right angles to each other at V.

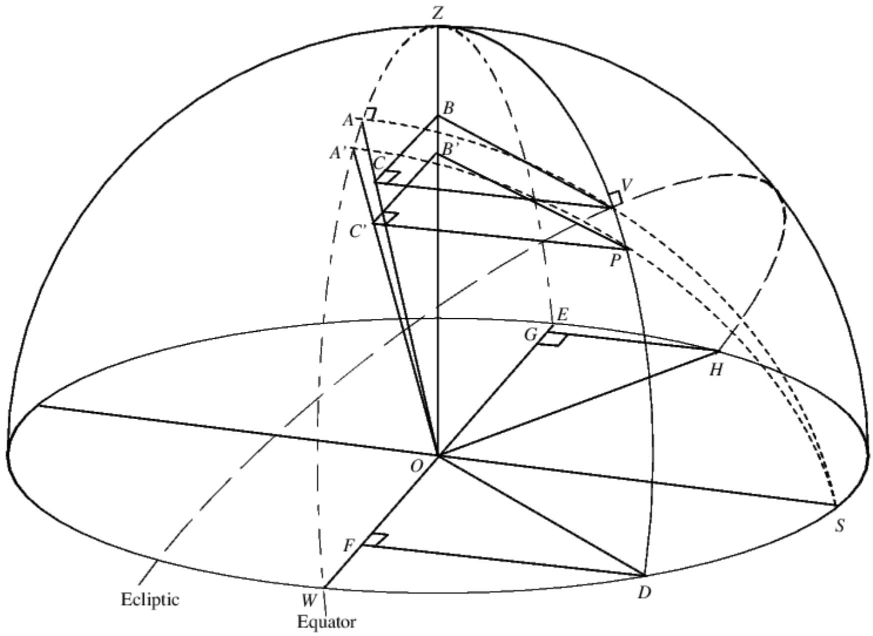

Figure 1: General view.

[3] This line has no connection with the main subject and we do not discuss it here.

$C'P$ is the Rsine of the declination of a planet P and CV the Rsine of the declination of nonagesimal V; and PU in Figure 3 is the Rsine of the latitude of P. The purpose of the formula is to compute $C'P$ using CV and PU.

However, the addition formula for the sine cannot be applied directly here because CV and PU are usually not on the same plane. Mādhava's application formula is essentially meant for handling this general situation.

Let ε be the maximum declination of the sun and $AV > PV$. Then, Mādhava's application formula can be expressed as:

$$\mathrm{Sin}A'P = \frac{\mathrm{Sin}AV \cdot \mathrm{Cos}PV}{R} \pm \frac{\mathrm{Sin}PV \cdot \mathrm{Cos}\varepsilon}{R}, \tag{4}$$

that is,

$$C'P = \frac{CV \cdot OU}{R} \pm \frac{PU \cdot \mathrm{Cos}\varepsilon}{R}. \tag{5}$$

The same two verses are cited also in Nīlakaṇṭha's *Tantrasaṅgraha*, [TaSa1977, ch. 6.4–5, p. 312]. Ramasubramanian and Sriram account for the formula using modern spherical trigonometry [TaSa2011, pp. 359–62].

Three calculations

Now we proceed to explain Nīlakaṇṭha's demonstration of the application formula in three stages.

Stage 1: Computing BV

The first step of the demonstration is to compute BV, the Rsine of the *dṛkkṣepa* ZV (see Figure 1), when CV and OG are given, which are respectively the Rsine of the declination of nonagesimal AV and the Rcosine of the amplitude of H, the point of intersection of the ecliptic and the horizon. Two different methods of computations are found.

Method 1: Here [the Rsine of] a *dṛkkṣepa* (BV = $\mathrm{Sin}ZV$) is obtained using the rule of three as follows [AB1957, p. 110, lines 18–19]:

सत्रिभग्रहद्युज्यया कोटिभूतया व्यासार्धतुल्यः कर्णो लभ्यते तदेष्टापक्रमज्यया कोट्या कियान् इति दृक्क्षेपो लभ्यते ।

satribhagrahadyujyayā koṭibhūtayā vyāsārdhatulyaḥ karṇo labhyate tadeṣṭāpakramajyayā koṭyā kiyān iti dṛkkṣepo labhyate |

When an hypotenuse equal to the radius ($HO = R$) is obtained from the R[co-]sine of the day [*dyujyā*, the radius of the diurnal circle of the planet] as a side, increased by three zodiacal signs ($GO =$ CosHE), then what is from the Rsine, as a side, of the given declination ($CV =$ SinAV), given the [Rsine of] *dṛkkṣepa*?

Method 2: Here [the Rsine of] a *dṛkkṣepa* ($BV{=}$SinZV) is obtained using the rule of three as follows [AB1957, p. 111, lines 19–21]:

ऊर्ध्वस्वस्तिकात् प्रभृति क्षितिजान्तं यद् दृक्क्षेपमण्डलव्यासार्धं तद् यदि कोटिस्वाहोरात्रार्धस्य कोटिरूपस्य कर्णः तदा ग्रहापक्रमतुल्यायाः कोट्याः कियान् कर्ण इति दृक्क्षेपो लभ्यते ।

ūrdhvasvastikāt prabhṛti kṣitijāntaṃ yad dṛkkṣepamaṇḍalavyāsārdhaṃ tad yadi koṭisvāhorātrārdhasya koṭirūpasya karṇaḥ tadā grahāpakramatulyāyāḥ koṭyāḥ kiyān karṇa iti dṛkkṣepo labhyate |

If the radius of the *dṛkkṣepamaṇḍala*, which begins from the zenith and ends at the horizon, is the hypotenuse of half [of the diameter] of the diurnal [circle] of the side [*koṭisvāhorātra*, FD] assuming the form of a side, then what is the hypotenuse corresponding to the side equal to [the Rsine of] the declination of the planet (CV)? [The result] gives the [Rsine of the] *dṛkkṣepa*.

These two computations eventually give the same result. Since, in Figures 1 and 2, two right triangles HGO and OFD are congruent and the BCV is similar to these two, we get two proportions

$$GO : HO :: CV : BV, \quad FD : OD :: CV : BV.$$

Accordingly we get;

$$BV = CV \cdot \frac{R}{GO} = CV \cdot \frac{R}{FD}. \tag{6}$$

Note: The phrase "the diurnal [circle] of the side (*koṭisvāhorātra*)" appearing in the method 2 can be understood as follows:

The radius of the diurnal circle of a planet with longitude λ is equal to the Rcosine of the latitude of the planet, $\mathrm{Cos}\delta_\lambda$, which is computed using the Rsine of the latitude $\mathrm{Sin}\delta_\lambda$:

$$\mathrm{Cos}\delta_\lambda = \sqrt{R^2 - \mathrm{Sin}^2\delta_\lambda}, \tag{7}$$

where $\mathrm{Sin}\delta_\lambda$ can be computed using a formula commonly used in Indian traditional astronomy:

$$\mathrm{Sin}\delta_\lambda = \mathrm{Sin}\lambda \cdot \frac{\mathrm{Sin}\varepsilon}{R}. \tag{8}$$

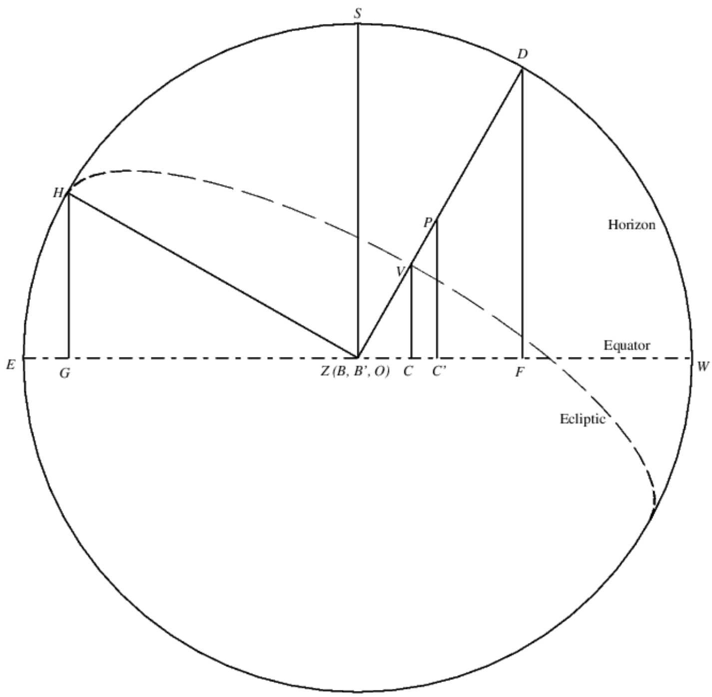

Figure 2: Looking down from the zenith.

From (7) and (8) it is noted that a 'normal' diurnal circle is computed using the Rsine of a longitude. Now, let us consider the diurnal circle of a point on the ecliptic the longitude of which equals the longitude of the planet increased by 90 degrees (three zodiacal signs). Because the Rsine of this point, $\mathrm{Sin}(\lambda+90°)$, is equal to the Rcosine of the longitude of the planet, $\mathrm{Cos}\lambda$, the radius of the diurnal circle of the point is computed as:

$$\begin{aligned}\mathrm{Cos}\delta_{\lambda+90^\circ} &= \sqrt{R^2 - \mathrm{Sin}^2\delta_{\lambda+90^\circ}} \\ &= \sqrt{R^2 - \left(\mathrm{Sin}(\lambda+90^\circ)\cdot\frac{\mathrm{Sin}\varepsilon}{R}\right)^2} \\ &= \sqrt{R^2 - \left(\mathrm{Cos}\lambda\cdot\frac{\mathrm{Sin}\varepsilon}{R}\right)^2},\end{aligned}$$

which shows that this radius of the diurnal circle is obtained from the Rcosine, or the *side*, of the longitude of the planet, $\mathrm{Cos}\lambda$; hence it is named "diurnal circle of the side." As far as this passage is concerned, the diurnal circle of the side of planet P is the diurnal circle of point H and its radius is $GO(=FD)$.

Stage 2: Applying the Sine addition formula

Consider Figure 3. Here, both BV and PU the Rsine of the latitude of planet P lie on the plane of the *dṛkkṣepamaṇḍala* (circle $ZVPD$). Hence we can use the addition formula for sine to get $B'P$, the Rsine of the arc ZP. Nīlakaṇṭha says [AB1957, p. 111, lines 23–24]:

> तत्र विक्षेपकोट्या दृक्क्षेपो हन्तव्यः । दृक्क्षेपकोट्या च विक्षेपः। पुनस्तद्योगः त्रिज्यया हर्तव्यः।
> *tatra vikṣepakoṭyā dṛkkṣepo hantavyaḥ | dṛkkṣepakoṭyā ca vikṣepaḥ | punas tadyogaḥ trijyayā hartavyaḥ |*
>
> In that case, [the Rsine of] the *dṛkkṣepa* ($BV = \mathrm{Sin}ZV$) should be multiplied by the side of the latitude ($OU = \mathrm{Cos}PV$), and [the Rsine of] the latitude ($PU = \mathrm{Sin}PV$) by the side of the *dṛkkṣepa*. Then the sum of those two should be divided by the radius (R).

That is,

$$B'P = \frac{BV \cdot OU + PU \cdot OB}{R}, \tag{9}$$

which comes straight from the addition formula.

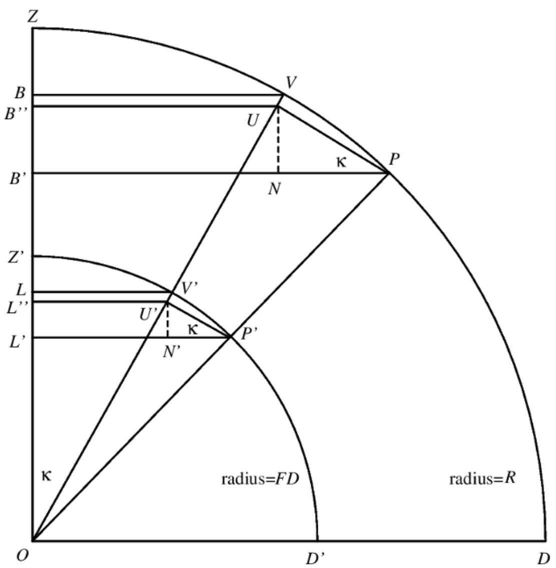

Figure 3: Two quarter circles.

Stage 3: Computing $C'P$

Then Nīlakaṇṭha computes $C'P$ from $B'P$ using another proportion similar to the proportion utilized for computing BV [AB1957, p. 111, line 24–p. 112, line 1]:

तेन कर्णेन तत्कोट्यानयने त्रैराशिकम् एवम्। यदि खमध्यात् प्रभृति दृक्क्षेपमण्डलगतया व्यासार्धतुल्यया जीवया ग्रहकोटिस्वाहोरात्रतुल्या कोटिर्लभ्यते तदा दृक्क्षेप[विक्षेप] चापयोगज्यया कर्णभूतया कियतीति। सैव स्फुटापक्रमज्या।

tena karṇena tatkoṭyānayane trairāśikam evam | yadi khamadhyāt prabhṛti dṛkkṣepa-maṇḍalagatayā vyāsārdhatulyayā jīvayā grahakoṭisvāhorātratulyā koṭir labhyate tadā dṛkkṣepa[vikṣepa][4] *cāpayogajyayā karṇabhūtayā kiyatīti | saiva sphuṭāpakramajyā |*

The rule of three for deriving the side ($C'P$) from the hypotenuse mentioned above ($B'P$) is as follows: if a side equal to [half the diameter of] the diurnal [circle] of the side of the planet ($B'P$) is obtained from the Rsine on the *dṛkkṣepamaṇḍala* beginning with the mid-heaven, which is equal to the radius (OD=R), then what is from the hypotenuse ($B'P$) [equal to] the Rsine of the sum of the *dṛkkṣepa* (ZV) [and the latitude (PV)]? The [result] is the Rsine of the true declination ($C'P$).

In Figure 1, because the two right triangles OFD and $B'C'P$ are similar,

$$\begin{aligned} OD : FD &:: B'P : C'P, \\ C'P &= B'P \cdot \frac{FD}{R}. \end{aligned} \tag{10}$$

Thus we get $C'P$, the Rsine of the true declination of the planet P.

The result combining the three stages

As the final step, Nīlakaṇṭha combines the results obtained in the three stages described above as follows [AB1957, p. 112, lines 1–6]:

तत्र त्रयाणां कर्मणाम् एकीकरणं प्रथमचरमयोर्मध्यमकर्मण्येव विलापनेन।

तत्र प्रथमत्रैराशिके त्रिज्या गुणकारः। कोटिस्वाहोरात्रार्धं भागहारः। द्वितीये पुनः स्फुटापक्रमकर्णस्य कोटिस्वाहोरात्रार्धं गुणकारः। व्यासार्धं भागहारः।

तत्र प्रथमत्रैराशिकसिद्धस्य दृक्क्षेपस्य विक्षेपकोटिर्गुणकारः। व्यासार्धं भागहारः। फलं विक्षेपेण संयोज्यम्।

[4] Insert *vikṣepa*.

tatra trayāṇāṃ karmaṇām ekīkaraṇaṃ prathamacaramayor madhyamakarmaṇy eva vilāpanena |[5]

tatra prathamatrairāśike trijyā guṇakāraḥ | *koṭisvāhorātrārdhaṃ bhāgahāraḥ* | *dvitīye punaḥ sphuṭāpakramakarṇasya koṭisvāhorātrārdhaṃ guṇakāraḥ* | *vyāsārdhaṃ bhāgahāraḥ* |

tatra prathamatrairāśikasiddhasya dṛkkṣepasya vikṣepakoṭir guṇakāraḥ | *vyāsārdhaṃ bhāgahāraḥ* | *phalaṃ vikṣepeṇa saṃyojyam* |

Then, the three computations are united by fusing the first (equation (6)) and the last (equation (10)) into the middle computation (equation (9)).

In that case, in the first rule of three (equation (6)), [for CV] the radius (R) is the multiplier (*guṇakāra*) and the diurnal circle of the side (FD) is the divisor (*bhāgahāra*). And in the second [rule of three] (equation (10)), for the hypotenuse corresponding to the true declination ($B'P$), the diurnal circle of the side is the multiplier and the radius is the divisor.

In that case, for [the Rsine of] the *dṛkkṣepa* (BV), the side of the latitude (OU) is the multiplier, the radius is the divisor. And the result should be added to [the result of another rule of three applied to the Rsine of] the latitude (PU).

The three equations, (6), (9) and (10), are here called 'the first', 'the middle computation' and 'the last' respectively, and, at the same time, equations (6) and (10) are also called 'the first rule of three' and 'the second (rule of three)' respectively, because they signify computations based on the rule of three.

Substituting equation (6) for BV in equation (9), and equation (9) for $B'P$ in equation (10), we get

$$\begin{aligned} C'P &= \left(\frac{CV \cdot \frac{R}{FD} \cdot OU}{R} + \frac{PU \cdot OB}{R}\right) \cdot \frac{FD}{R} \\ &= CV \cdot \frac{R}{FD} \cdot \frac{OU}{R} \cdot \frac{FD}{R} + PU \cdot \frac{OB}{R} \cdot \frac{FD}{R} \\ &= CV \cdot \frac{OU}{R} + PU \cdot \frac{OB}{R} \cdot \frac{FD}{R}. \end{aligned} \tag{11}$$

In order to show that (11) is the same as the formula given by Mādhava (5), we need to show that the second term of the right side of equation (11) is the same as that of equation (5).

$$PU \cdot \frac{OB}{R} \cdot \frac{FD}{R} = \frac{PU \cdot \text{Cos}\varepsilon}{R}.$$

[5] The reading has been amended to *vilāpanena* for *nilāpanena*.

Nīlakaṇṭha proceeds to do this first by transforming $\frac{OB}{R}$ as follows [AB1957, p. 112, lines 16–19]:

तत्र व्यासार्धे भागहारे दृक्क्षेपकोटिर्गुणकारः। कोटिस्वाहोरात्रे भागहारे को गुणकार इति दृक्क्षेपकोट्या ग्रहकोट्यपक्रमस्वाहोरात्रस्य च घाताद् व्यासार्धेन हृतं फलं गुणकारः। तच्च परमापक्रमकोटितुल्यम्।

tatra vyāsārdhe bhāgahāre dṛkkṣepakoṭir guṇakāraḥ | koṭisvāhorātre bhāgahāre ko guṇakāra iti dṛkkṣepakoṭyā grahakoṭyapakramasvāhorātrasya ca ghātād vyāsārdhena[6] *hṛtaṃ phalaṃ guṇakāraḥ | tac ca paramāpakramakoṭitulyam |*

In that case, [considering that], 'because the side of the *dṛkkṣepa* (OB) is the multiplier (for the Rsine of the latitude (PU)) when the radius is the divisor, what is the multiplier if the diurnal circle of the side (FD) is the divisor,' [we find that] the multiplier is the product of the side of the *dṛkkṣepa* and [the radius of] the diurnal circle corresponding to the declination of the side of the planet (*grahakoṭyapakramasvāhorātra*, FD), divided by the radius. This is equal to the side of the maximum declination ($\mathrm{Cos}\varepsilon$).

Nīlakaṇṭha seems to find "the multiplier" for the Rsine of the latitude (PU) when the diurnal circle of the side (FD) is assumed to be the divisor. Using the relation,

$$\frac{OB}{R} = \frac{\text{multiplier}}{FD},$$

he should reach the conclusion that

$$\text{multiplier} = \frac{OB \cdot FD}{R}. \tag{12}$$

Then Nīlakaṇṭha states clearly that the multiplier equals to the side of the maximum declination, $\mathrm{Cos}\varepsilon$. He demonstrates it in the following part by showing that the square of "the multiplier" equals to $FD^2 - CV^2$ [AB1957, p. 112, line 20–p. 113, line 5]:

तत्र व्यासार्धस्य भागहारत्वे दृक्क्षेपो भुजा तयोः कोटिर्गुणकारः| कोटिस्वाहोरात्रस्य भागहारत्वे अपक्रम एव भुजा। तद्भुजानयन एवं त्रैराशिकम्। व्यासार्धकर्णस्य दृक्क्षेपो भुजा तदा कोटिस्वाहोरात्रकर्णस्य का भुजेति। पूर्वम् अपक्रमाद् एव दृक्क्षेप आनीतः। एतद्विपरीतम् एव तत्कर्म। तत्र त्रैराशिकत्रये प्रथमत्रैराशिकेन दृक्क्षेप आनीयते। तत्रापक्रमस्य व्यासार्धं गुणकारः। कोटिस्वाहोरात्रार्धं भागहारः। तद्व्यत्ययेन दृक्क्षेपाद् अपक्रमानयने। तस्मात् कोटिस्वाहोरात्रस्य भागहारत्वे तद्व्यासार्धवृत्तगता भुजा दृक्क्षेपस्थानीया अपक्रम एव। तस्माद् अपक्रमवर्गं कोटिस्वाहोरात्रवर्गाद् विशोध्यापि मूलीकृतं यत् सैवेह कोटिः। तस्या एव गुणकारत्वम् अपि।

tatra vyāsārdhasya bhāgahāratve dṛkkṣepo bhujā tayoḥ koṭir guṇakāraḥ | koṭisvāhorātrasya bhāgahāratve apakrama eva bhujā | tadbhujānayana evaṃ trairāśikam | vyāsārdhakarṇasya dṛkkṣepo bhujā tadā koṭisvāhorātrakarṇasya kā

[6] The term '*dyuvyāsardhena*' has been amended to '*vyāsārdhena*'.

bhujeti | pūrvam apakramād eva dṛkkṣepā ānītaḥ | etadviparītam eva tatkarma | tatra trairāśikatraye prathamatrairāśikena dṛkkṣepa ānīyate | tatrāpakramasya vyāsārdhaṃ guṇakāraḥ | koṭisvāhorātrārdhaṃ bhāgahāraḥ | tadvyatyayena dṛkkṣepād apakramānayane | tasmāt koṭisvāhorātrasya bhāgahāratve tadvyāsārdhavṛttagatā bhujā dṛkkṣepasthānīyā apakrama eva | tasmād apakramavargaṃ koṭisvāhorātravargād viśodhyāpi mūlīkṛtaṃ yat saiveha koṭiḥ | tasyā eva guṇakāratvam api |

In that case, when the radius (OV) is the divisor, [the Rsine of] the *dṛkkṣepa* (BV) is the arm, and the side corresponding to these two (OB) is the multiplier. When [the radius of] the diurnal circle of the side (FD) is the divisor, [the Rsine of] the declination (CV) is the arm.

The rule of three for deriving such arms is as follows: when [the Rsine of] the *dṛkkṣepa* (BV) is the arm corresponding to the radius as an hypotenuse, then what is the arm corresponding to [the radius of] the diurnal circle of the side as an hypotenuse?

[The Rsine of] the *dṛkkṣepa* (BV) was previously derived only from [the Rsine of] the declination (CV); that computation [mentioned here] is opposite to it. In the previous case, [the Rsine of] the *dṛkkṣepa* is computed using the first among the three rules of three. There, for [the Rsine of] the declination the radius is the multiplier and the diurnal circle of the side is the divisor. When computing [the Rsine of] the declination from [the Rsine of] the *dṛkkṣepa*, the inverse [procedure is applied].

Therefore when the diurnal circle of the side (FD) is the divisor, the arm on the circle whose radius is it (FD)—which should be substituted for [the Rsine of] the *dṛkkṣepa* (BV) [when the radius (OV) is the divisor]—is [the Rsine of] the declination (CV). Then the square root of the square of [the radius of] the diurnal circle of the side (FD) diminished by the square of [the Rsine of] the declination (CV) is exactly the side. And it is also the multiplier.

In Figure 3 the radius of the larger quarter circle is R, and that of the smaller one equals FD. Because two right triangles BOV and LOV' are similar,

$$OV : BV :: OV' : LV'$$
$$OV : OB :: OV' : OL.$$

Hence,

$$LV' = \frac{BV \cdot OV'}{OV} = \frac{BV \cdot FD}{R} \tag{13}$$
$$OL = \frac{OB \cdot OV'}{OV} = \frac{OB \cdot FD}{R}. \tag{14}$$

Comparing equation (13) to equation (6), we find

$$LV' = CV. \tag{15}$$

And also comparing equation (14) and equation (12), we find that OL is nothing but “the multiplier”. Therefore in the triangle in LOV',

$$OL^2 = OV'^2 - LV'^2,$$

that is, multiplier$^2 = FD^2 - CV^2$, or

$$\text{multiplier} = \sqrt{FD^2 - CV^2}. \tag{16}$$

Next, Nīlakaṇṭha states that the square of FD is equal to $R^2 - HG^2$ [AB1957, p. 113, lines 6–7].

> कोटिस्वाहोरात्रार्धस्य वर्गो व्यासार्धवर्गाद् ग्रहकोट्यपक्रमवर्गरहितः।
>
> *koṭisvāhorātrārdhasya vargo vyāsārdhavargād grahakoṭyapakramavargarahitaḥ* |
>
> The square of the radius of the diurnal circle of the side (FD) equals the square of the radius (R) diminished by the square of the declination for the side of the planet (HG).

As demonstrated above, the two right triangles, HGO and OFD, in Figures 1 and 2 are congruent with each other. Therefore,

$$FD^2 = R^2 - OF^2 = R^2 - HG^2.$$

After this Nīlakaṇṭha demonstrates that "the Multiplier" equals Cosε [AB1957, p. 113, lines 8–11]:

> कोट्यपक्रमवर्गस्य भुजापक्रमवर्गस्य च योगः परमापक्रमवर्ग एव परमापक्रमज्यावृत्तगतत्वात् तयोः। तस्मात् व्यासार्धपरमापक्रमयोर्वर्गविश्लेषमूलम् एव गुणकारः। तस्यैव कोटिस्वाहोरात्रकर्णस्य ग्रहापक्रमभुजायाश्च कोटित्वात्।
>
> *koṭyapakramavargasya bhujāpakramavargasya ca yogaḥ paramāpakramavarga eva paramāpakramajyāvṛttagatatvāt tayoḥ* | *tasmāt vyāsārdhaparamāpakramayor vargaviśleṣamūlam eva guṇakāraḥ* | *tasyaiva koṭisvāhorātrakarṇasya grahāpakramabhujāyāś ca koṭitvāt* |
>
> The sum of the square of [the Rsine of] the declination for the side of the planet (HG) and the square of [the Rsine of] the declination as an arm (CV) equals the square of [the Rsine of] the maximum declination (Sinε), for both of them are on the circle of the Rsine of the maximum declination. Therefore, the square root of the difference between the squares of the radius (R) and of [the Rsine of] the maximum declination is the multiplier, for it is exactly the side corresponding to [the radius of] the diurnal circle of the side (FD) as the hypotenuse and to [the Rsine of] the planet's declination (CV) as the arm.

Nīlakaṇṭha mentions here that

$$HG^2 + CV^2 = \text{Sin}^2\varepsilon. \tag{17}$$

As a reason for this, Nīlakaṇṭha just says, "both of them are on the circle of the Rsine of the maximum declination." This means that they can be assumed to be on a circle of radius $\mathrm{Sin}\varepsilon$ so that they might have an arm-side-hypotenuse relation (of a right-angled triangle) with the radius $\mathrm{Sin}\varepsilon$, though he does not give any demonstration.

This can be demonstrated easily within the framework of traditional Indian astronomy. Let λ be the longitude of V and δ_λ be its declination. For CV, the Rsine of the declination of V ($\mathrm{Sin}\delta_\lambda$), and HG, the Rsine of the declination of the point H whose longitude equals $\lambda + 90°$ ($\mathrm{Sin}\delta_{(\lambda+90°)}$), we have two relations:

$$CV = \mathrm{Sin}\delta_\lambda = \frac{\mathrm{Sin}\lambda \cdot \mathrm{Sin}\varepsilon}{R},$$
$$HG = \mathrm{Sin}\delta_{(\lambda+90°)} = \frac{\mathrm{Sin}(\lambda+90°) \cdot \mathrm{Sin}\varepsilon}{R} = \frac{\mathrm{Cos}\lambda \cdot \mathrm{Sin}\varepsilon}{R}.$$

Therefore,

$$\begin{aligned} HG^2 + CV^2 &= \frac{\mathrm{Cos}^2\lambda \cdot \mathrm{Sin}^2\varepsilon}{R^2} + \frac{\mathrm{Sin}^2\lambda \cdot \mathrm{Sin}^2\varepsilon}{R^2} \\ &= \frac{\left(\mathrm{Cos}^2\lambda + \mathrm{Sin}^2\lambda\right) \cdot \mathrm{Sin}^2\varepsilon}{R^2} = \frac{R^2 \cdot \mathrm{Sin}^2\varepsilon}{R^2} = \mathrm{Sin}^2\varepsilon. \end{aligned}$$

Now, substituting (**??**) for FD^2 in (16),

$$\text{multiplier} = \sqrt{R^2 - (HG^2 + CV^2)}, \tag{18}$$

and substituting (17) for $HG^2 + CV^2$ in (18), he finally obtains

$$\text{multiplier} = \sqrt{R^2 - \mathrm{Sin}^2\varepsilon} = \mathrm{Cos}\varepsilon, \tag{19}$$

as required.

Finally completing the demonstration, Nīlakaṇṭha observes [AB1957, p. 113, lines 11–17]:

तस्मात् विक्षेपज्यां परमापक्रमकोट्या निहत्य कोटिस्वाहोरात्रेण हृत्वा पुनर्द्वितीयत्रैराशिके कोटिस्वाहोरात्रेण गुणनं त्रिज्यया हरणं च कार्यम्। तस्मात् कोटिस्वाहोरात्रेण गुणनं हरणं चोभयमपि न कर्तव्यम्। तस्मात् परमापक्रमकोटिर्गुणकारः। व्यासार्धं भागहारः। अत उक्तम्।

परमापक्रमकोट्या विक्षेपज्यां निहत्य तत्कोट्या इष्टक्रान्तिं चोभे त्रिज्याप्ते इति।

tasmād vikṣepajyāṃ paramāpakramakoṭyā nihatya koṭisvāhorātreṇa hṛtvā punar dvitīyatrairāśike koṭisvāhorātreṇa guṇanaṃ trijyayā haraṇaṃ ca kāryam | tasmād koṭisvāhorātreṇa guṇanaṃ haraṇaṃ cobhayam api na kartavyam | tasmād paramāpakramakoṭir guṇakāraḥ | vyāsārdhaṃ bhāgahāraḥ | ata uktam |

> *paramāpakramakoṭyā vikṣepajyāṃ nihatya tatkoṭyā iṣṭakrāntiṃ cobhe trijyāpte iti |*

Therefore the Rsine of the latitude is multiplied by the side of the maximum declination and divided by [the radius of] the diurnal [circle] of the side, and then at the second rule of three a multiplication with [the radius of] the diurnal circle of the side and a division with the radius (R) should be applied. Accordingly, the multiplication and division by [the radius of] the diurnal [circle] should not be done. Therefore, the side of the maximum declination is the multiplier and the radius is the divisor [for the Rsine of the latitude]. Therefore [the following rule] is stated [by Mādhava]:

> Multiply the Rsine of the latitude (PU) by the side of the maximum declination and the Rsine of given the declination by its side (OU), and divide those two by the radius (R).

Using equation (19), the second term of the right hand side of Nīlakaṇṭha's formula (11) can be rewritten as:

$$PU \cdot \frac{OB}{R} \cdot \frac{FD}{R} = PU \cdot \frac{\text{multiplier}}{FD} \cdot \frac{FD}{R} = PU \cdot \frac{\text{Cos}\varepsilon}{R},$$

which is the same as the second term of Mādhava's formula.

Thus it is demonstrated that Mādhava's application formula (5) is equivalent to equation (11), and is an exact result.

Deriving formula (11) in another way

Subsequently, Nīlakaṇṭha shows another way for deriving equation (11) [AB1957, p. 113, line 22–p. 114, line 7]:

तस्मादेवमेव दृक्क्षेपमण्डलेन राशिकूटद्वयबद्धेन मार्गेणापक्रममण्डलाद्विक्षिप्तस्य ग्रहस्य तद्विक्षेपस्य तिर्यक्त्वात् घटिकामण्डलापेक्षया ऋजुत्वम् आनेयम्। ऋजुत्वं च कोटिस्वाहोरात्रगुणनत्रिज्याहरणाभ्यां स्यात्।

पुनस्तस्यापक्रमस्य कोटिस्वाहोरात्रवृत्तगतत्वात् तद्योगवियोगयोः "जीवे परस्पर" इत्याद्युक्तकर्मणा क्रियमाणयोः स्ववृत्तकोट्यानयने स्वस्ववर्गं कोटिस्वाहोरात्रवर्गाद्विशोध्य शेषस्य मूलीकरणं कार्यम्। तयोरितरस्य कोट्या द्वौ भुजौ गुणयित्वा कोटिस्वाहोरात्रेणैव हृत्वा योगो वियोगो वा कार्यः। एतदेव प्रथमतः प्राप्तं कर्म। तत्र विक्षेपस्य ऋजूकरणायापक्रमयोगवियोगयोग्यत्वाय च ये त्रैराशिके तयोरेकीकरणमेवेह प्रदर्शितम्।

tasmād evam eva dṛkkṣepamaṇḍalena rāśikūṭadvayabaddhena mārgeṇāpakrama-maṇḍalād vikṣiptasya grahasya tadvikṣepasya tiryaktvād ghaṭikāmaṇḍalāpekṣayā ṛjutvam āneyam | ṛjutvaṃ ca koṭisvāhorātraguṇanatrijyāharaṇābhyāṃ syāt |

punas tasyāpakramasya koṭisvāhorātravṛttagatatvāt tadyogaviyogayoḥ 'jīve paras-para' ityādyuktakarmaṇā kriyamāṇayoḥ svavṛttakoṭyānayane svasvavargaṃ koṭi-svāhorātravargād viśodhya śeṣasya mūlīkaraṇaṃ kāryam | tayor itarasya koṭyā dvau bhujau guṇayitvā koṭisvāhorātreṇaiva hṛtvā yogo viyogo vā kāryaḥ | etad eva pratha-mataḥ prāptaṃ karma | tatra vikṣepasya ṛjūkaraṇāyāpakramayogaviyogayogyatvāya ca ye trairāśike tayor ekīkaraṇam eveha pradarśitam |

Therefore when the planet (P) deflects from the declination circle, [the ecliptic] along the *dṛkkṣepamaṇḍala*, because its latitude has obliqueness (*tiryaktva*) with regard to the equator, its straightness (*ṛjutva*) should be computed in the same way.

The straightness can be obtained [from the Rsine of the latitude (PU)] by means of multiplication by [the radius of] the diurnal [circle] of the side (FD) and division by the radius (R).

On the other hand, because [the Rsine of] the declination of the [planet] (CV) is on the diurnal circle of the side, [its straightness does not need to be computed].

When their sum and difference are computed using [the formula] mentioned [by Mādhava] beginning with "two Rsines mutually (*jīve paraspara*)", for deriving sides on the circle of themselves [i.e., the diurnal circle of the side], having subtracted their squares separately from the square of [the radius of] the diurnal circle of the side, compute the square root of the remainder. Having multiplied two arms by the other sides of those two, divide them by [the radius of] the diurnal circle of the side, and add them up or take the difference. This is the computation obtained at the beginning. For that case, the calculation for uniting two rules of three—[one is] for making the latitude straight and [the other is] for making the declination appropriate for addition and subtraction—has been shown here.

With the word 'straightness' (*ṛjutva*), Nīlakaṇṭha seems to express the component of the Rsine of the latitude (PU) parallel [7] to the Rsine of the declination of V (i.e., CV), which is computed as follows:

$$\text{straightness}_{(PU)} = PU \cdot \frac{FD}{R}. \tag{20}$$

This is equivalent to the calculation for converting PU on the *dṛkkṣepamaṇḍala* (circle $ZVPD$) onto the diurnal circle of the side (circle $Z'V'P'D'$). The two quarter circles in Figure 3 represent two such circles. These two circles are actually in different planes but $P'U'$ represents straightness$_{(PU)}$. Furthermore, we have the relation, $CV = LV'$, as already shown in (15). LV' and $P'U'$ are the Rsines of the two arcs $Z'V'$ and $V'P'$, respectively, on the diurnal circle. On the other hand, their Rcosines can be computed as follows:

[7] Śrīdhara too uses the word *ṛjutva* in this sense. See [Hay2013, pp. 245–332].

$$OL = \sqrt{FD^2 - LV'^2},$$
$$OU' = \sqrt{FD^2 - P'U'^2}.$$

By means of these relationships, the Rsine of the sum of $Z'V'$ and $V'P'$, i.e., $L'P'$, can be computed just as in the case of (3),

$$\begin{aligned} L'P' &= L'N' + N'P' \\ &= LV' \cdot \frac{OU'}{FD} + P'U' \cdot \frac{OL}{FD}. \end{aligned}$$

Then, this $L'P'$ is reduced to the *dṛkkṣepamaṇḍala*. The result is $B'P$ in Figure 3. Furthermore, $C'P$, the Rsine of the declination of the planet P, is computed from $B'P$ by using equation (10). Combining these two equations, we obtain,

$$C'P = B'P \cdot \frac{FD}{R} = L'P' \cdot \frac{R}{FD} \cdot \frac{FD}{R} = L'P'.$$

Thus, it turns out that $L'P'$ is equal to $C'P$. Accordingly, we have the relation,

$$C'P = LV' \cdot \frac{OU'}{FD} + P'U' \cdot \frac{OL}{FD}.$$

Then, applying to this relation equation (20) and the two proportional relations,

$$\frac{OU'}{FD} = \frac{OU}{R} \quad \text{and} \quad \frac{OL}{FD} = \frac{OB}{R},$$

which can be easily obtained from Figure (3), and substituting CV for LV', we finally arrive at the equation,

$$C'P = CV \cdot \frac{OU}{R} + PU \cdot \frac{FD}{R} \cdot \frac{OB}{R},$$

which is identical with Nīlakaṇṭha's formula (11).

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Astronomical instruments in Bhāskarācārya's *Siddhāntaśiromaṇi*

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1 Introduction

In his most comprehensive and critical history of *Bhāratīya Jyotiṣa*, the great savant Sankara Balakrishna Dikshit remarks that Bhāskarācārya used his exceptional talents in formulating rationales for the mathematical and astronomical procedures but not in advancing the techniques of observational astronomy [Dik1981, pp. 114–123, esp. 120–121]. Indeed, even in Bhāskarācārya's treatment of the conventional astronomical instruments, one misses the magisterial assurance with which he deals with other topics and notices instead ambivalence between what is practical and what is speculative.

Bhāskarācārya devotes an exclusive chapter to the description of several astronomical instruments in his *Siddhāntaśiromaṇi*, yet he dismisses them in favour of a straight piece of stick which he calls *Dhī-yantra*. Even so, he proudly declares that he has invented a new instrument called *Phalaka-yantra* which he introduces with a special *maṅgalācaraṇa* in the middle of the chapter. He derides the dimensions prescribed in the earlier texts for the water clock (*Ghaṭikā-yantra*) as illogical (*yukti-śūnya*) and impractical (*durghaṭa*), but does not hesitate to dwell on different types of perpetual motion machines (*Svayaṃvaha-yantras*), although he himself was doubtful about their relevance in observational astronomy.

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This ambivalence, this oscillation between what is practical and what is speculative, characterizes the *Yantrādhyāya* which Bhāskara included in the *Golādhyāya* of his *Siddhāntaśiromaṇi*, following the footsteps of Brahmagupta. So we retrace five centuries backwards and begin the narration with Brahmagupta.

1.1 Astronomical instruments prior to Bhāskara

In the twenty-second chapter of the *Brāhmasphuṭasiddhānta*, called appropriately *Yantrādhyāya*, Brahmagupta offers the first systematic and comprehensive account of astronomical instruments (*kāla-yantras*) [Sar2008a, pp. 47–63]. The astronomical instruments described by him are *Dhanus, Turyagola, Cakra, Yaṣṭi, Śaṅku, Ghaṭikā, Kapāla, Kartarī* and *Pīṭha*. In addition, he describes some varieties of automata and perpetual motion machines (*Svayaṃvaha-yantra*) [BSS1902].[1] The armillary sphere (*Gola*), however, receives an exclusive treatment in the twenty-first chapter entitled *Golādhyāya*.

This procedure was emulated by subsequent writers like Lalla in the eighth century and Śrīdhara in the eleventh. In his *Śiṣyadhīvṛddhida*, Lalla devotes the fifteenth chapter *Golabandhādhikāra* for the construction of the armillary sphere and the twenty-first chapter *Yantrādhyāya* for the following instruments: some types of automata, *Gola, Cakra, Dhanus, Kartarī, Kapāla, Pīṭha, Bhagaṇa, Śaṅku, Ghaṭī, Śalākā, Śakaṭa* and *Yaṣṭi*. Of these *Bhagaṇa, Śalākā* and *Śakaṭa* are new; these have not been described by Brahmagupta.

In his *Siddhāntaśekhara*, Śrīpati discusses the instruments somewhat briefly as compared to Lalla. The construction of the armillary sphere is described in the fifteenth chapter entitled *Golādhyāya* (vv. 29–39). The nineteenth chapter entitled *Yantrādhyāya* deals with the following nine instruments: *Svayaṃvaha-gola-yantra, Cakra, Dhanus, Kartarī, Kapāla, Pīṭha, Śaṅku, Ghaṭī* and *Yaṣṭi*.

1.2 Instruments discussed by Bhāskara

Closely following these three predecessors, Bhāskarācārya devotes, in the *Golādhyāya* of his *Siddhāntaśiromaṇi*, an exclusive chapter entitled *Golabandhādhikāra* to the armillary sphere and a subsequent chapter called *Yantrād-*

[1] Besides these, Brahmagupta also describes eight types of accessories (*saṃsādhan*); cf. [Sar2008e].

hyāya to ten different types of measuring instruments and three varieties of perpetual motion machines. The instruments discussed by him in the *Yantrādhyāya* are as follows: *Gola, Nāḍīvalaya, Ghaṭikā, Śaṅku, Cakra, Cāpa, Turya, Phalaka, Yaṣṭi* and *Dhī*. In addition, he discusses *Nalaka-yantra* in the *Tripraśnādhikāra* of the *Grahagaṇita*.

Moreover, while discussing the *Nāḍīvalaya-yantra*, he cites one and half verses from his own description of a *Sarvatobhadra-yantra (yathā mayā sarvatobhadra-yantre paṭhitaḥ)*.[2] It implies clearly that he invented the *Sarvatobhadra-yantra* and described it. We do not know whether he composed an exclusive treatise on the *Sarvatobhadra-yantra* or whether he discussed the instrument in some other text. Nevertheless, it is intriguing why Bhāskara, aside from this reference *en passant*, did not include in the present *Yantrādhyāya* a full description of the construction and use of the *Sarvatobhadra-yantra* which is obviously his own invention like the *Phalaka-yantra*.

Bhāskara discards some instruments discussed by his predecessors and adds some of his own. Those discarded are *Kapāla, Kartarī, Pīṭha* of Brahmagupta and *Śakaṭa* and *Śalākā* of Lalla. Those introduced newly by Bhāskara are *Nāḍīvalaya, Phalaka* and *Dhīyantra*.

Lancelot Wilkinson and his protégé Bapudeva Sastri prepared an excellent translation of the *Golādhyāya*, including the *Yantrādhyāya*, with useful notes and diagrams [SL1861]; the late Professor R. N. Rai described the construction and use of these instruments competently in a special volume of the *Indian Journal of History of Science*; [Rai1985, pp. 308–336] and Yukio Ôhashi discussed the same very thoroughly in his doctoral thesis, a part of which is published in the *Indian Journal of History of Science* [Oha1994, pp. 155–314].

Therefore, we shall not go into the construction and use of these instruments in detail; our focus will be rather on Bhāskara's attitude towards observational astronomy in general and towards certain instruments in particular. We shall also discuss the antecedents of the instruments described by Bhāskara and touch upon where it is necessary, the echoes of Bhāskara's writings in other texts.

[2] The verses on the *Sarvatobhadra-yantra* will be discussed below in 4.3.

2 *Gola-yantra* (Armillary sphere)

2.1 Antecedents

As stated above, Brahmagupta discusses the *Gola-yantra* exclusively in the twenty-first chapter of his *Brāhmasphuṭasiddhānta* and the other instruments in the twenty-second chapter named *Yantrādhyāya*. The reason for this separate and exclusive treatment of the *Gola-yantra* is that this instrument is considered an essential part of the *Gola* (Spherics), which deals with a host of imaginary circles in the heavens such as the celestial equator, the ecliptic and so on, and teaches methods to determine the linear and angular distances between them. By arranging these imaginary circles as physical entities in the armillary sphere, the student can clearly grasp the purpose and function of these imaginary circles. Therefore, *gola-bandha*, i.e., putting together different rings to constitute the armillary sphere, is a primary task in understanding the *Gola*. It is for this reason that Bhāskara I, who is Brahmagupta's contemporary, also discusses the construction of the armillary sphere quite elaborately at the beginning of his commentary on the *Gola-pāda* of the *Āryabhaṭīya* [Lu2015, pp. 1–19].

The armillary sphere was discussed also by Greek astronomers, including Ptolemy who describes its construction in his *Almagest* [Pto1984, ch. 5.1, pp. 217–219]. His armillary sphere, which he calls *astrolabon*, consists of six rings. There is a fixed meridian ring inside which the ecliptic ring and the solstitial colure are fitted. Two latitude rings are attached, one outside the solstitial colure and the other inside it. Inside the inner ring is added another ring which is equipped with two sighting holes at diametrically opposite points. Armillary spheres made according to Ptolemy's model with minor additions can be seen in nearly every museum in Europe. Arab astronomers followed Ptolemy's model, but not a single specimen of Arabic armillary sphere (*dhāt al-ḥalaq*) seems to have survived.[3] But the *Gola-yantra* described in Sanskrit texts is quite different from Ptolemy's instrument. As Ôhashi remarks, the main purpose of the *Gola-yantra* is to ascertain the equatorial coordinates, while Ptolemy's armillary seeks to determine ecliptic coordinates (cf. [Oha1994, p. 272]). Moreover, the *Gola-yantra* is much more elaborate than Ptolemy's armillary sphere. For example, in the *Gola-yantra* envisaged

[3] In the early eighteenth century, Sawai Jai Singh caused Ptolemy's description of astronomical instruments to be rendered in Sanskrit through Naṣīr al-Dīn al-Ṭūsī's Arabic version of the *Almagest* and collected these and similar other descriptions in the *Yantraprakāra*; later on he got the entire *Almagest* translated into Sanskrit. For the Sanskrit rendering of Ptolemy's description, see [Sar1986, pp. 16–18, 54–58].

by Brahmagupta, there are as many as 51 movable rings besides several fixed rings.[4]

In the *Golādhyāya* chapter Brahmagupta defines the various circles, their mutual relationships and so on, but does not dwell on the practical aspects of the construction of the *Gola-yantra*, [BSS1902, ch. 21.49–58] which is done by Lalla in his *Śiṣyadhīvṛddhida*.

2.2 Lalla on the Gola-yantra

Lalla arranges the various rings in three groups called *kha-gola* (sphere of the sky), *bha-gola* (sphere of the fixed stars) and *graha-gola* (sphere of the planets). Each group or assemblage of rings constitutes a separate shell or sphere and these are arranged one inside the other at three slightly different levels on the polar axis (*Dhruva-yaṣṭi*). The *kha-gola* represents horizontal system of coordinates. It consists of six rings which stand for the prime vertical (*sama-maṇḍala*), the prime meridian (*yāmyottara-maṇḍala*), the horizon (*kṣitija-maṇḍala*), two e*koṇa*-circles (two great circles, one passing through the zenith, the south-east and the south-west points, and the other through the zenith, the south-west and the north-east points), and the six o'clock circle (*unmaṇḍala*). All these rings are marked with 360 degrees.

The *bha-gola* (sphere of the fixed stars) represents the equatorial and ecliptic coordinates. It consists of the equator (*viṣuvadvṛtta*, also called *nāḍī-vṛtta*) which is divided into 60 *ghaṭikās*, the solstitial colure, ecliptic (*apamaṇḍala*), six *dyu-maṇḍalas* (diurnal circles), three each on either side of the equator, corresponding to the six pairs of signs [ŚiDh1981b, ch. 15.1–15] (cf. [Oha1994, pp. 268–269]).

The *graha-gola* is the sphere of the planets, where diverse rings are fixed for each planet. At the end of the chapter, Lalla enumerates all the movable circles in the armillary sphere, as has been done by Brahmagupta previously [ŚiDh1981b, ch. 15.31–32].

मन्दोच्चनीचवृत्तानि सप्त पञ्च शीघ्रवृत्तानि ।
प्रतिमण्डलान्यपि तथा दृग्दृक्षेपापवलयानि ॥
प्रत्येकमिनादीनां चन्द्रादीनां विमण्डलानि च षट् ।
पञ्चाशदेकयुक्ता चलवृत्तानां भवत्येवम् ॥

mandoccanīcavṛttāni sapta pañca śīghravṛttāni |
pratimaṇḍalānyapi tathā dṛgdṛkṣepāpavalayāni ||

[4] *calavṛttāny ekapañcāśat* [BSS1902, ch. 21.69].

pratyekam inādīnāṃ candrādīnāṃ vimaṇḍalāni ca ṣaṭ |
pañcāśadekayuktā calavṛttānāṃ bhavatyevam ||

There are seven epicycles, seven eccentrics, seven *dṛṅ-maṇḍalas* (circles passing through the planet and zenith), seven *dṛkkṣepa-maṇḍalas* (secondaries to the ecliptic passing through the zenith) and seven *apamaṇḍalas* (ecliptics), one for each planet beginning with the sun. Then, there are five *śīghra* epicycles and five *śīghra* eccentrics, one for each of the five planets. There are also six *vimaṇḍalas* (planetary orbits), one for each planet beginning with the moon. Thus, [altogether] there are 51 movable circles.[5]

2.3 Bhāskara on the Gola-yantra

Bhāskara made some changes to Lalla's design, by introducing *dṛg-gola*, in addition to *bha-gola* and *kha-gola*. The *bha-gola* is the sphere of the fixed stars. Instead of having a separate *bha-gola* for each planet, only one *bha-gola* is used for all planets. It consists of the ecliptic, equinoctial, diurnal circles, etc., which are movable. It is fixed to the polar axis so that it may move freely by moving the axis. Beyond this sphere is fixed the *kha-gola* or the celestial sphere; it consists of the prime vertical, meridian, horizon etc., which remain fixed in a given latitude. Beyond these two is the *dṛg-gola* in which the circles forming both the spheres *kha-gola* and *bha-gola* are combined.

At the beginning of the chapter entitled *Golabandhādhikāra*, Bhāskara explains the construction of the *Gola-yantra* in a nutshell [SiŚi1981, *Golabandhādhikāra* v. 2]:

कृत्वादौ ध्रुवयष्टिमिष्टतरुजामृज्वीं सुवृत्तां ततो
यष्टीमध्यगतां विधाय शिथिलां पृथ्वीमपृथ्वीं बहिः ।
बध्नीयाच्छशिसौम्यशुक्रतपनारेज्यार्किभानां दृढान्
गोलांस्तत्परितः श्लथौ च नलिकासन्ध्यौ खदृग्गोलकौ ॥

kṛtvādau dhruvayaṣṭim iṣṭatarujām ṛjvīṃ suvṛttāṃ tato
yaṣṭīmadhyagatāṃ vidhāya śithilāṃ pṛthvīṃ apṛthvīṃ bahiḥ |
badhnīyācchaśisaumyaśukratapanārejyārkibhānāṃ dṛḍhān
golāṃs tatparitaḥ ślathau ca nalikāsandhyau khadṛggolakau ||

First, he should prepare a straight and well-rounded polar axis from [the timber of] any desired tree. He should then attach loosely a small [sphere to represent the] earth in the middle of the axis. Beyond it, he should affix firmly the spheres of the Moon, Mercury, Venus, the Sun, Mars, Jupiter, Saturn and of the fixed stars [successively one encompassing the other]. Around these, he should add two loose spheres of *kha-gola* and *dṛg-gola*, fastened by means of small tubes.

[5] In the last line, Chatterjee's edition reads *ca vṛttānāṃ*; it should actually read *calavṛttānāṃ*. Her translation has been modified accordingly from 'great circles (which are not fixed)' to 'movables circles'.

Having discussed the construction and use of the *Gola-yantra* in the *Gola-bandhādhikāra*, Bhāskara teaches how to determine the ascendant (*lagna*) with the help of the *Gola-yantra* at the beginning of the *Yantrādhyāya* [SiŚi1981, *Yantrādhyāya* vv. 3–5ab].

2.4 Extant specimens

Since the time of Brahmagupta, if not earlier, teachers of astronomy must have taught their pupils how to prepare a *Gola-yantra* with strips of bamboo. None of these have survived. However, there are extant a few recent metal specimens. The full details of these instruments will appear in my *Descriptive Catalogue of Indian Astronomical Instruments*, which is under preparation. Here follows a brief overview:

1 In the Maharaja's Sanskrit College at Mysore, there is a *Gola-yantra* designed by Mahāvidvān Karur Seshachar according to the teachings of Bhāskarācārya.[6] This is a unique piece because here can be seen the entire range of rings envisaged by Bhāskarācārya. Its dimensions are not known.

2 In the 1830s, Lancelot Wilkinson saw a *Gola-yantra* and several other instruments on the top of the fort at Kota in Rajasthan. Wilkinson thought that these were set up at the time of Rao Umaid Singh, who ruled Kota from 1771 to 1819 [Wil1834, p. 515].

   At present this *Gola-yantra* and the other instruments are displayed in the Palace Museum of Kota [see Figure 2]. Virendra Nath Sharma, who has studied these instruments, reports that the *Gola-yantra* consists of three spheres of rings, with diameters measuring 120, 111, 99.5 cm respectively [Sha2000, pp. 233–244].

3 In the Linden Museum at Stuttgart, Germany, there is an armillary sphere containing two sets of rings around a wooden axis. The outer ring measures 57 cm, the inner ring 50 and the wooden axis 80 cm. It is said to be from the palace of the Maharaja of Gwalior.

4 Another *Gola-yantra* is with the Royal Museum of Scotland at Edinburgh. Its inner diameter measures 85 cm. There are rings to represent the ecliptic, equator, tropics, and meridian. Twelve great circles intersect the ecliptic at right angles, passing through its zodiacal divisions. What distinguishes this large specimen is the large number of star-pointers. Twenty

[6] I am grateful to Dr. James McHugh, University of Southern California, for sending me a photograph of this globe.

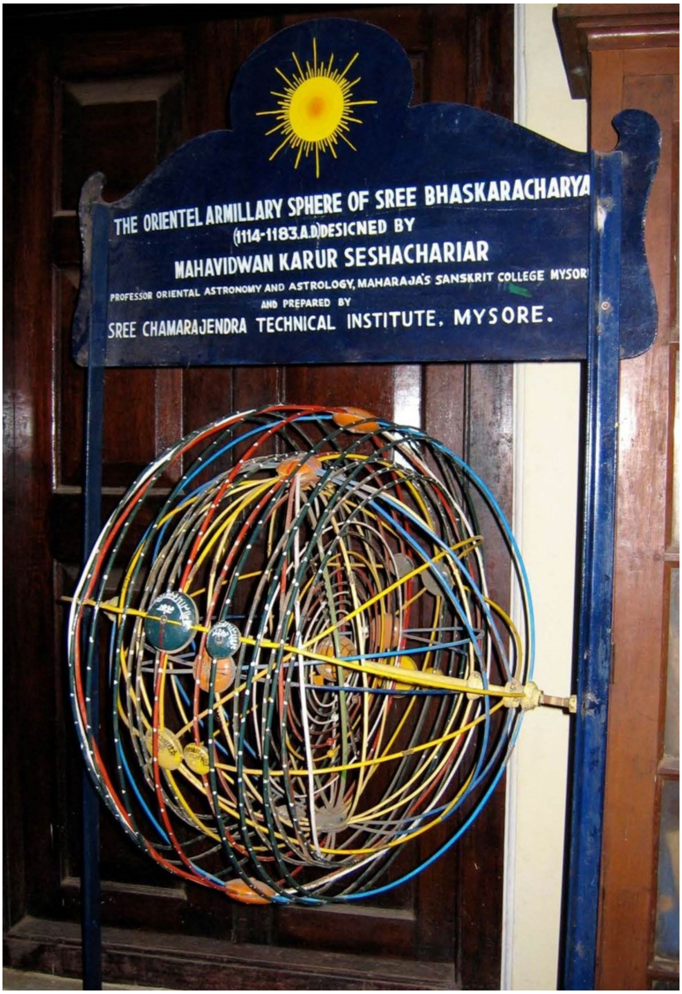

Figure 1: *Gola-yantra* made according to the description of Bhāskarācārya Maharaja's Sanskrit College, Mysore. (Photo courtesy of Dr. James McHugh).

named star-pointers are attached to the rings. Positions of several more stars are marked on the rings. Its provenance is not known.

5 Jai Singh Observatory, Jaipur, holds a small armillary sphere with a diameter of 53 cm and consisting of eight rings.

6 The Sanskrit Mahāvidyālaya, M. S. University of Baroda at Vadodara owns a simple *Gola-yantra* consisting of seven rings. It was made at the beginning of the twentieth century along with a few other instruments for use in the classes of *Jyotiḥśāstra* [Sar2009, pp. 14–16].

Figure 2: *Gola-yantra*, Palace Museum, Kotah. (Photo courtesy of Dr. Alexander Walland, Ingelheim, Germany).

3 *Ghaṭikā-yantra* (Water clock)

3.1 Antecedents of the Ghaṭikā-yantra

In his *Science and Civilisation in China*, Joseph Needham classifies the water clocks into three distinct types: inflow, outflow and the sinking bowl.[7] The second and the third types were used in India, but not the inflow type. The outflow type was mentioned as early as in the *Vedāṅga-jyotiṣa*. [VJ1985, p. 37 (Y-VJ 24; R-VJ 17)] and later on in the *Arthaśāstra* [ArŚā2010, ch. 1.19.6].

Sometime about the fifth century CE, this outflow water clock was replaced by the sinking bowl type.[8] It consists of a hemispherical bowl made of copper with a small aperture at the bottom. When it is placed on the surface of water in a larger basin, water enters the bowl from below through the aperture, fills

[7] [Nee1965, vol. 3, p. 315]. For explanatory drawings of these three types, see [Oha1994, p. 273, figure 52].

[8] However, the outflow water clock continued to be used in some automata, as shown below in section 6.

Figure 3: *Gola-yantra*, Royal Museum of Scotland, Edinburgh. (Photo by S. R. Sarma).

it, and makes it sink. The weight of the bowl and the size of the aperture are so adjusted that the bowl sinks in 24 minutes. In Sanskrit, the bowl is called *ghaṭikā/ghaṭī*, 'a small pot'. The time unit measured by this device also came to be called *ghaṭikā/ghaṭī*. The entire ensemble was called *ghaṭikā-yantra/ghaṭī-yantra* or simply *ghaṭikā/ghaṭī* [Sar2008c].

Although the instruments for measuring time had changed, the standard unit of time remained the same duration of 24 minutes which was known both by the older set of names *nāḍikā/nāḍī* or *nālikā/nālī* as well by the new set

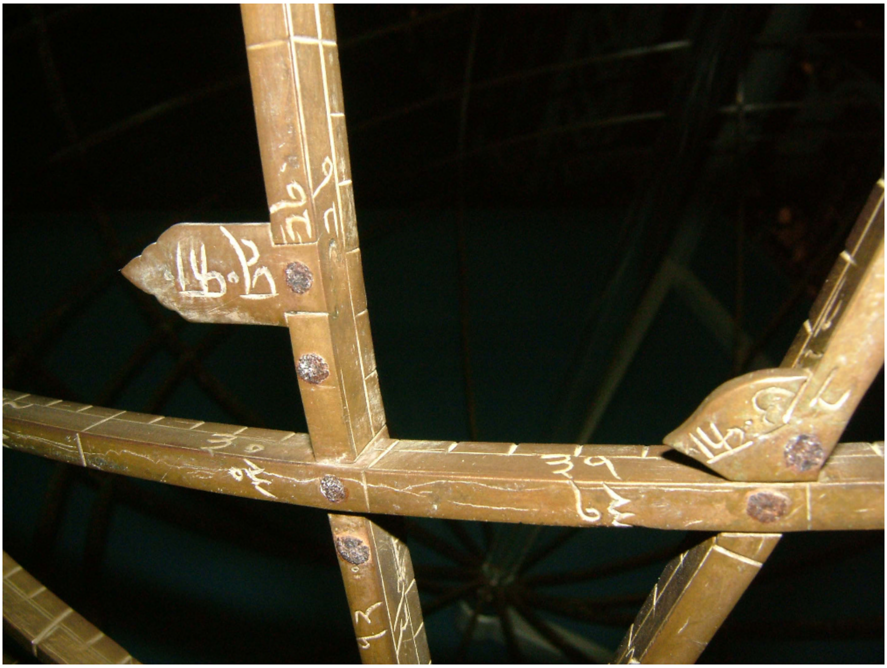

Figure 4: Detail of the Armillary Sphere in the figure above. (Photo by S. R. Sarma). Besides the divisions marked on the rings, two star pointers named *U[ttarā] Phā[lgunī]* and *P[ūrvā] Phā[lgunī]* can be seen here.

ghaṭikā/ghaṭī. It is not unusual that in the same text names from both the sets are employed.

The sinking bowl water clock was described for the first time by Āryabhaṭa in his *Āryabhaṭasiddhānta* in the early sixth century;[9] thereafter it was described in several astronomical and other texts. Interestingly enough, of all the traditional instruments, the water clock is the only one for which the astronomers strove to prescribe exact measurements, not only for the bowl but also for the aperture at the bottom of the bowl. For instance, Lalla describes it thus in his *Śiṣyadhīvṛddhida* [ŚiDh1981b, ch. 21.34–35]:

दशभिः शुल्बस्य पलैः पात्रं कलशार्धसन्निभं घटितम् ।
हस्तार्धमुखव्यासं समघटवृत्तं दलोच्छ्रायम् ॥
सत्र्यंशमाषकत्रयकृतनलया समवृत्तया हेम्नः ।
चतुरङ्गुलया विद्धं मज्जति विमले जले नाड्या ॥

[9] This work is now lost, but the descriptions of various instruments survive in Rāmakṛṣṇa Ārādhya's commentary on the *Sūryasiddhānta*; cf. [Shu1967]. See also [Sar2004b].

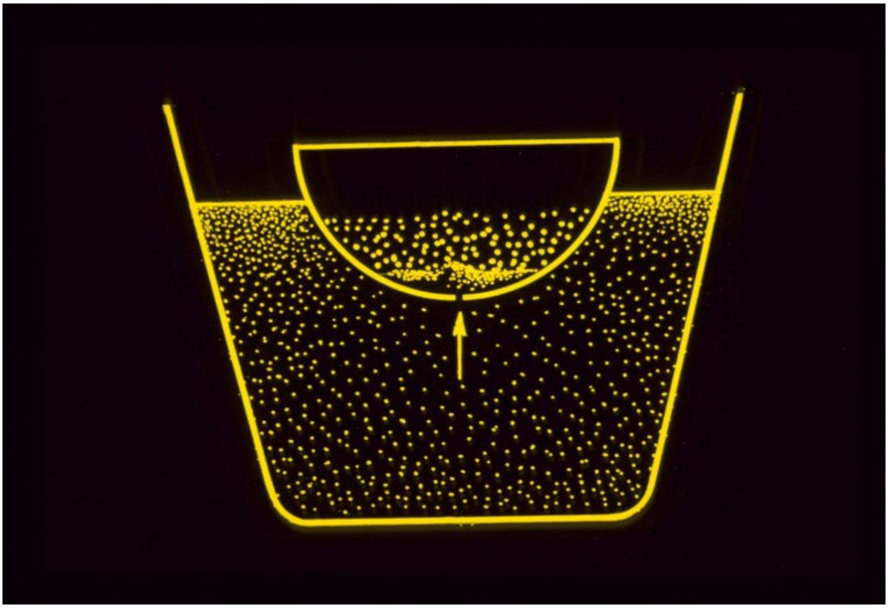

Figure 5: Sinking Bowl Type of Water Clock.

daśabhiḥ śulbasya palaiḥ pātraṃ kalaśārdhasannibhaṃ ghaṭitam |
hastārdhamukhavyāsaṃ samaghaṭavṛttaṃ dalocchrāyam ||
satryaṃśamāṣakatrayakṛtanalayā samavṛttayā hemnaḥ |
caturaṅgulayā viddhaṃ majjati vimale jale nāḍyā ||

The hemispherical bowl made of ten *palas* of copper, which is half a cubit (i.e. twelve *aṅgulas*) in diameter at the mouth and half (i.e. six *aṅgulas*) as high, which is evenly circular, and which is bored by a uniformly circular needle (*nalā*), made of three and one-third *māṣas* of gold and of four *aṅgulas* in length, sinks in clear water in one *ghaṭikā (nāḍī).*

Obviously a thin gold needle cannot bore an aperture into a copper bowl. What Lalla means is that the aperture should be such that a gold needle of given dimensions can just pass through it. That is to say, Lalla defines the very minute aperture in terms of the cross-section of gold wire weighing $3\frac{1}{3}$ *māṣas* which is stretched to a length of 4 *aṅgulas*. Goldsmiths can no doubt draw fine grades of gold wire, but it would be impossible to draw a wire measuring exactly eight *aṅgulas* from a lump of gold weighing exactly $3\frac{1}{3}$ *māṣas*. It is doubtful if this prescription was ever followed in practice.

3.2 Bhāskara on the Ghaṭikā-yantra

Therefore Bhāskara disapproves of this specification and declares [SiŚi1981, *Yantrādhyāya* ch. 8, pp. 441–442]:

घटदलरूपा घटिता घटिका ताम्री तले[ऽ]पृथुच्छिद्रा ।
द्युनिशनिमज्जनमित्या भक्तं द्युनिशं घटीमानम् ॥

ghaṭadalarūpā ghaṭitā ghaṭikā tāmrī tale [']pṛthucchidrā |
dyuniśanimajjanamityā bhaktaṃ dyuniśaṃ ghaṭīmānam ||

A copper bowl (*tāmrī ghaṭikā*), formed like a hemisphere (*ghaṭa-dala-rūpā*), having a small (*apṛthu*) aperture at the bottom. [The duration of] a day and night divided by the number of immersions [of this bowl] in a day and night is the measure of the water clock (*ghaṭī-mānam*).

This statement is elaborated in his *Vāsanābhāṣya* as follows [SiŚi1981, *Vāsanābhāṣya* ch. 8, p. 442]:

अत्र दशभिः शुल्बस्य पलैरित्यादि यद्घटीलक्षणं कैश्चित् कृतं तद्युक्तिशून्यं दुर्घटं चेत्येतदुपेक्षितम्।

atra daśabhiḥ śulbasya palair ityādi yad ghaṭīlakṣaṇaṃ kaiścit kṛtaṃ tad yuktiśūnyaṃ durghaṭaṃ cetyetad upekṣitam |

Here, we ignore the definition of the water clock given by certain [scholars like Lalla] as *daśabhiḥ śulbasya palair* etc., because it is illogical (*yukti-śūnya*) and difficult to implement (*durghaṭa*).

From the practical point of view Bhāskara's critique of Lalla's specifications is quite justified. On the other hand, the alternative offered by Bhāskara is likewise impractical, when he suggests that a bowl with any duration of immersion will do. If, for example, the bowl sinks in 34 minutes, then it measures roughly 1.416 *ghaṭikās*. How can such a bowl be used to measure time in terms of *ghaṭikās* of 24 minutes without resorting to tedious computations? Therefore, the duration of the water clock has to be one *ghaṭikā* or its multiples and not any arbitrary period.

Bhāskara's predecessors, therefore, wisely prescribe that the bowl should be such that it immerses 60 times in an *ahorātra*; in other words, it measures exactly one *ghaṭikā*. Thus Āryabhaṭa, besides prescribing exact dimensions for the bowl and its perforation, provides also a simple alternative when he adds that the bowl (irrespective of its dimensions) should sink sixty times in a day and night [Shu1967, p. 95]:

स्वेष्टं वान्यदहोरात्रे षष्ट्याम्भसि निमज्जति ।
ताम्रपात्रमधश्छिद्रम् अम्बुयन्त्रं कपालकम् ॥

sveṣṭaṃ vānyad ahorātre ṣaṣṭyāmbhasi nimajjati |
tāmrapātram adhaśchidram ambuyantraṃ kapālakam ||

Alternatively (*vā*), any hemispherical (*kapāla*) copper vessel made according to one's liking with an aperture in the bottom, which sinks in water sixty times in a day and night, is the [accurate] water clock (*ambu-yantra*)

Brahmagupta does not prescribe any dimensions whatsoever, but says merely that bowl should sink sixty times in an *ahorātra* [BSS1902, ch. 22.1]:

घटिका कलसार्धाकृति ताम्रं पात्रं तलेऽपृथुच्छिद्रम् ।
मध्ये तज्जलमज्जनषष्ट्या द्युनिशं यथा भवति ॥

ghaṭikā kalasārdhākṛti tāmraṃ pātraṃ tale'pṛthucchidram |
madhye tajjalamajjanaṣaṣṭyā dyuniśaṃ yathā bhavati ||

The *Ghaṭikā-yantra* is a copper vessel of the shape of a hemisphere. At the center of the bottom is a small aperture so made that the bowl sinks sixty times in a day and night.

The *Sūryasiddhānta* also gives a similar definition [SūSi1959, ch. 13.23]:

ताम्रपात्रमधश्छिद्रं न्यस्तं कुण्डेऽमलाम्भसि ।
षष्टिर्मज्जत्यहोरात्रे स्फुटं यन्त्रं कपालकम् ॥

tāmrapātram adhaśchidraṃ nyastaṃ kuṇḍe'malāmbhasi |
ṣaṣṭirmajjatyahorātre sphuṭaṃ yantraṃ kapālakam ||

If a copper vessel having an aperture in the bottom, when placed in a basin of pure water, sinks sixty times in a day and night, then it is an accurate hemispherical instrument.

By not insisting that the bowl should sink in one *ghaṭikā* as his predecessors did, Bhāskara also underestimates the Indian artisan's ability to fabricate bowls of exactly one *ghaṭikā* duration. The few extant specimens with different sizes and shapes (see Figure 6 below) show that the artisans did not follow the measurements given by the astronomers. Nevertheless, they were able to make by empirical methods bowls that sank in a prescribed duration of time.

There is yet another problem with Bhāskara's statement. His wording (*ghaṭadala-rūpā ghaṭitā ghaṭikā tāmrī tale'pṛthucchidrā*) follows very closely that of Brahmagupta (*ghaṭikā kalaśārdhākṛti tāmraṃ pātraṃ tale'pṛthucchidram*). Since the manuscripts rarely mark the *avaghraha* symbols indicating the *'a'* which is elided owing to *sandhi*, these have to be inserted according to the context by the modern editors or readers.[10] In the present case, we have to assume that Brahmagupta and Bhāskara consider the aperture to be *'apṛthu'*, for the simple reason that a large aperture will make the bowl sink very quickly. For example, if it sinks in ten minutes, then the bowl has to be lifted, emptied and placed again on the surface of the water every ten minutes!

[10] [SL1861, p. 211] also read *pṛthu* and translate as 'A *ghaṭī* made of copper like the lower half of a water-pot, should have a large hole bored in its bottom.'

Figure 6: Water clock bowls from India (Kerala, Tamilnadu and Uttar Pradesh), Burma and Sri Lanka. The smaller one in the foreground is from Kerala and is made of a coconut shell. Pitt Rivers Museum of Ethnology, Oxford. (Photo by S. R. Sarma).

There is no practical advantage in using a water clock of such short duration. Or, to use Bhāskara's own language, it would be illogical (*yukti-śūnya*) to measure time with a water clock of such short duration. The aperture has to be very small so that the bowl does not sink too quickly. It is for this reason that several texts lay emphasis on the smallness of the aperture by defining its size in terms of the cross-section of a hypothetical gold wire of certain weight and length, although it would be impossible to construct the bowl according to this prescription as we have shown above. Moreover, all the extant water clocks contain very small apertures.[11]

[11] The sinking bowl type of water was used throughout India (and also in South Asia and South-East Asia) until the end of the nineteenth century for telling time. Baden Powell [Pow1872, p. 200] reports thus: 'This article is in common use, and by it all police guards, &c, keep the time, striking their gong as each hour comes round.' One would therefore expect scores of water clocks to be surviving in every part of India. But people usually recycle unused copper or brass vessels and therefore there are few specimens extant. The largest collection of seven bowls is preserved in the Pitt Rivers Museum of Ethnology at Oxford (see Figure 6). The bowl in the upper left was used in the guard-rooms at Mirzapur.

3.3 Munīśvara on the aperture

In his commentary *Vāsanāvārttika* on the *Siddhāntaśiromaṇi*, Nṛsiṃha is silent on the question whether the aperture should be *pṛthu* or *apṛthu*, but another commentator Munīśvara (b. 1603) forcefully argues in favour of the attribute *pṛthu* [SiŚi1952, p. 367]:

> तलेऽधोभागे पृथुच्छिद्रा महारन्ध्रा घटिका कार्या। एतेन जलपूर्णपात्रे निक्षिप्ताधश्छिद्रे जलागमने जलान्तर्गतमलादिकं वस्तु प्रतिबन्धकं न भवति तथा छिद्रं कार्यमिति सूचितम्। सूक्ष्मच्छिद्रे मलादिकेन तत्प्रतिबन्धसम्भवात् अकारप्रश्लेषो न युक्तः।
>
> *tale'dhobhāge pṛthucchidrā mahārandhrā ghaṭikā kāryā | etena jalapūrṇapātre nikṣiptādhaśchidre jalāgamane jalāntargatamalādikaṃ vastu pratibandhakaṃ na bhavati tathā chidraṃ kāryam iti sūcitam | sūkṣmacchidre malādikena tatpratibandhasambhavād akārapraśleṣo na yuktaḥ |*
>
> The bowl should be so made that it has a large aperture (*pṛthucchidra* = *mahārandhra*) at the bottom. Through this statement it is indicated that the aperture should be made in such a manner that, when it is placed on the water of the basin and when water enters [the bowl], the aperture is not blocked by any dirt which might be in the water of the basin. Because of the possibility of the small aperture getting blocked by dirt and the like, inserting here the vowel *a* (*akāra-praśleṣa*) [by reading *apṛthu*] is not warranted.

Obviously Munīśvara did not come across any actual specimens in Varanasi in the seventeenth century or did not pay attention to them. As for his fear that a small aperture can be blocked by dirt in the water, the texts usually prescribe that the water in the basin should be pure. For example, in the verses just cited above, the *Sūryasiddhānta* mentions *amalāmbhasi* and the *Śiṣyadhīvṛddhida* stipulates *vimale jale*.

The rest of the instruments can be divided into two groups of linear and circular instruments.

4 Linear instruments

4.1 Śaṅku (Gnomon)

Like the water clock, the instrument called *Śaṅku* (gnomon) was also in use in India for a long time. In its simplest form, the *Śaṅku* consists of a straight stick stuck into the earth vertically in order to determine the time of the day by the position of its shadow. The *Siddhānta* texts teach the employment of the gnomon for ascertaining the three elements, namely, the points of compass

(*dik*), the observer's latitude (*deśa*) and the time (*kāla*) in the chapter on the *Tripraśna* [Oha1994, pp. 168–196]. Bhāskara also discusses *Śaṅku* in the *Tripraśnādhikāra* quite extensively. In the *Yantrādhyāya*, he merely adds that the gnomon should be made of ivory and that it should be turned on a lathe so that the circumference at the top and bottom remains the same [SiŚi1981, *Yantrādhyāya* v. 9].

4.2 Yaṣṭi (Staff)

The next three instruments—*Yaṣṭi* (staff), *Nalaka* (tube) and *Dhī* (intellect)—are variations of the gnomon. In the case of the *Yaṣṭi*, one draws a circle on a level ground, marks the cardinal points on it and employs the simple staff as the representation of the Radius of the celestial sphere to determine the position of heavenly bodies [SiŚi1981, pp. 28–39]. The *Yaṣṭi* is also used in terrestrial surveys for determining the heights and distances of objects.

4.3 Nalaka (Sighting tube)

The *Nalaka-yantra* is almost the same as the *Yaṣṭi* with the difference that, while the *Yaṣṭi* is made of solid bamboo, the *Nalaka* is a hollow bamboo tube. This is not a measuring instrument per se, but a device for verifying the correctness of computation. In the *Tripraśnādhikāra* of his *Śiṣyadhīvṛddhida*, Lalla enjoins that the computed planetary positions be verified by means of a hollow tube (*nalaka*) mounted upon two bamboo poles erected cross-wise on the ground so that the lower end of the tube is at the observer's eye-level and the observer can view the planet whose position has been previously determined by computation [ŚiDh1981b, ch. 4.47–48]. The same procedure is adopted by Śrīpati in his *Siddhāntaśekhara* [SiŚe1932, ch. 4.84–85].

Bhāskara also discusses the use of the *Nalaka* in the *Tripraśnādhikāra* (vv. 105–109) but introduces a new note. He states that after making the necessary preparations, the planet should be shown through the *Nalaka-yantra* to the benevolent king for his entertainment.[12] 'After making the necessary

[12] दर्शयेद्दिविचरं के वानेहसि द्युचरदर्शनयोग्ये ।
पूर्वमेव विरचय्य यथोक्तं रञ्जनाय सुजनस्य नृपस्य ॥
darśayed divicaraṃ ke vānehasi dyucaradarśanayogye |
pūrvam eva viracayya yathoktaṃ rañjanāya sujanasya nṛpasya ||
[SiŚi1981, *Tripraśnādhikāra* v. 109]

arrangements as told before, the planet should be shown to the benevolent king for his amusement, either [in the sky or its reflection] in water at the moment which is suitable for viewing the planet.' This is the only occasion when Bhāskara makes a reference to a king, although several of his ancestors (and also descendants) enjoyed royal patronage.

The sighting tube is akin to the telescope without lenses; even so, it can be used with advantage in viewing the objects in the sky. It appears to have been very popular in India. There exist several unpublished manuscripts exclusively devoted to the sighting tube such as Rāmakṛṣṇa's *Nalikābandha-krama-paddhati* (Maharaja Man Singh II Museum, Jaipur), Vāmanaprasāda's *Nalakayantrasādhana* (Chandra Shumshere Collection, Bodleian Library, Oxford), Gopirāja's *Nalikāprabandha* (Rajastan Oriental Research Institute, Jodhpur) and so on.

In the Sanskrit Mahāvidyālaya of the M. S. University of Baroda, there is a modern version of *Nalaka-yantra* which was produced at the beginning of the twentieth century for use in the Jyotiṣa classes. Unlike the *Nalaka-yantra* described in Sanskrit texts, this one is equipped to measure the altitude and azimuth of heavenly bodies like the theodolite. However, its stand and mounting are different from that of the theodolite.

4.4 Dhī-yantra (Intelligence instrument)

After describing several instruments, some briefly and some at length, Bhāskara dismisses all of them in favour of a straight piece of stick which he designates as the *Dhī-yantra* [SiŚi1981, *Yantrādhyāya* vv. 40–41].

अथ किमुपृथुतन्त्रैर्धीमतो भूरियन्त्रैः
स्वकरकलितयष्टेर्दत्तमूलाग्रदृष्टेः ।
न तदविदितमानं वस्तु यद्दृश्यमानं
दिवि भुवि च जलस्थं प्रोच्यतेऽथ स्थलस्थम् ॥
वंशस्य मूलं प्रविलोक्य चाग्रं तत्स्वान्तरं तस्य समुच्छ्रयं च ।
यो वेत्ति यष्ट्यैव करस्थयासौ धीयन्त्रवेदी वद किं न वेत्ति ॥

atha kimupṛthutantrairdhīmato bhūriyantraiḥ
svakarakalitayaṣṭerḥ dattamūlāgradṛṣṭeḥ |
na tad aviditamānaṃ vastu yad dṛśyamānaṃ
divi bhuvi ca jalasthaṃ procyate'tha sthalastham ||
vaṃśasya mūlaṃ pravilokya cāgraṃ tat svāntaraṃ tasya samucchrayaṃ ca |
yo vetti yaṣṭyaiva karasthayāsau dhīyantravedī vada kiṃ na vetti ||

What does a man of intelligence want with an array of instruments on which heavy tomes are written? For him who holds a staff (*yaṣṭi*) in his hand and casts his eye

Figure 7: *Nalaka-yantra* with a stand. On the base plate can be seen a spirit-level and circular frame for the compass, which is missing. (Photo by S. R. Sarma).

> from its base to the top there is no object in sight of which he cannot find out the measurements in the sky as well as on the earth, be it in water or on dry ground.
>
> The knower of the *Dhī-yantra* is one who can determine the distance and height of a bamboo, after observing its root and crown by means of the staff in his hand. Tell me what else is there that he cannot measure!

But the staff alone cannot provide linear or angular distances unless its projection is drawn on the level ground. Bhāskara goes on to state that by holding the staff in such a manner that it points to the pole star and by the projections of the two ends of the staff on the circle drawn on the level ground, one obtains the equinoctial midday shadow (*palabhā*) and with it one computes several parameters. He also teaches how to use the stick for terrestrial measurements. But it cannot replace all the diverse kinds of instruments discussed by him and by other astronomers. As Ôhashi remarks quite rightly Bhāskara clearly overrates this instrument.[13] The astute Nṛsiṃha Daivajña is rather baffled by this instrument and remarks as follows in his *Vāsanāvārtika* commentary [SiŚi1981, p. 472]:

> सर्वत्र यन्त्रनिर्माणं यन्त्रे वेधप्रकारश्चोच्यते। अत्र करकलितयष्टिरेव यन्त्रनिर्माणम्। दत्तमूलाग्रदृष्टिरेव वेधप्रकारः। धीमत एव करकलितयष्ट्या वांशौच्च्यादिज्ञानं संभवति नान्यस्येति धीयन्त्रमिदम् इति स्पष्टम्।
>
> *sarvatra yantranirmāṇaṃ yantre vedhaprakāraścocyate | atra karakalitayaṣṭireva yantranirmāṇam | dattamūlāgradṛṣṭireva vedhaprakāraḥ | dhīmata eva karakalitayaṣṭyā vaṃśauccyādijñānaṃ saṃbhavati nānyasyeti dhīyantram idam iti spaṣṭam* |
>
> In the case of every [other] instrument, the method of construction and the method of observation are stated. Here holding the staff in the hand is itself the method of construction. Looking at the base and the top of the staff is the method of observation. Only an intelligent person, and no other, can know the height and so on of a bamboo, by holding the staff in his hand. Thus it is evident that it is called 'Intelligence Instrument.'[14]

[13] [Oha1994, p. 236]: 'As Bhāskara says, the staff gives enough information for Hindu astronomy, if one spares no pains to do complicated calculations. However, it does not mean that other convenient instruments are unnecessary. It seems that Bhāskara II overpraised the *dhī-yantra*.'

[14] According to Powell [Pow1872, p. 201] *Buddhi-yantra*, which is another name for *Dhī-yantra*, was displayed at the Lahore Exhibition of 1864. Powell describes it as 'a large gnomon the shadow of which is observed when the sun is in the equinoctial.' But Bhāskara envisaged its use not just 'when the sun is in the equatorial,' but at all times when the sun is shining.

5 Circular instruments

5.1 Cakra, Cāpa and Turya

The three instruments, *Cakra* (circle), *Cāpa* or *Dhanus* (semi-circle) and *Turyagola* or *Turya* (quadrant), are closely related in shape and function. *Cakra* is a circular wooden plate with its circumference graduated into 360 degrees, *Dhanus* is its half, and *Turya-gola* the quarter. In all the three, a perforation is made at the centre into which a thin rod is inserted as the axis and a plumb-line is suspended from the centre. These instruments are so held towards the sun that the axis throws a shadow on the circumference. Then the arc intercepted between the nadir (indicated by the plumb-line) and the shadow is the zenith-distance. Brahmagupta prefers the semi-circular variety and explains all functions in connection with *Dhanus* and adds that the same can be done with *Turyagola* and *Cakra*.[15] But his successors show a marked preference for the *Cakra*, presumably because it has the ideal shape.

Bhāskara explains in detail how to prepare the circular disc: its rim is to be marked with 360 degrees and vertical and horizontal diameters are to be drawn on it; at the end of the horizontal diameter should be marked the horizon and at the end of the vertical diameter the zenith. A thin rod should be affixed at the centre of this disc as its axis. With this instrument one can measure the sun's altitude (*unnatāṃśa*) and the zenith-distance (*natāṃśa*) [SiŚi1981, *Yantrādhyāya* vv. 10–12].

Bhāskara also explains how to measure the longitude of a planet at night (*graha-vedha*) [SiŚi1981, *Yantrādhyāya* vv. 13–15]. For this, the disc should be held in the plane of the ecliptic. This is done as follows: Keeping the eye at the lower part of the rim, one tilts the disc so that any two fixed stars with zero latitude are visible above the upper rim. Since the stars with zero latitude are situated on the ecliptic, the disc is now in a plane parallel to the ecliptic. In this connection Bhāskara mentions the stars *paitryarkṣa* = *maghā* (α Leonis, Regulus), *puṣya* (δ Cancri), *antima* = *revatī* (ζ Piscium) and *vāruṇa* = *śatatārakā* (λ Aquarii). If these are not visible on the sky at the given time, one should make use of stars with minimum latitude such as *rohiṇī* (α Tauri).

Holding the disc in this position firmly, one should mark the position of one of the two stars on the rim. Then the angular distance between the star and the desired planet should be measured on the rim. This distance, increased

[15] [BSS1902, ch. 22: vv. 8–16 (*Dhanus*), v. 17 (*Turyagola*), v. 18 (*Cakra*)] .

or diminished by the longitude of the star (according as the planet is to the west or east of the star), gives the longitude of the planet (*sphuṭa-graha*).

Having described the construction and use of the *Cakra-yantra* in detail, Bhāskara adds laconically that the *Cāpa* and *Turyagola* are a half and a quarter respectively of the *Cakra* and that all the tasks of *Cakra* can be performed with *Cāpa* and *Turya-gola* as well. However, the *Turya-gola* or the quadrant is a handier device and played an important role in India in the period after Bhāskara.

5.2 Nāḍīvalaya

A variant of the *Cakra-yantra* is the *Nāḍīvalaya* which is an equatorial sundial. It was first described by Lalla under the name *Bhagaṇa*. [ŚiDh1981b, ch. 21.27–30]. Bhāskara changed the name to *Nāḍīvalaya* because the circular plate rests in a plane parallel to the celestial equator which is called *nāḍīvalaya* in Sanskrit [SiŚi1981, *Yantrādhyāya* vv. 5cd–7]. The *Nāḍīvalaya-yantra* consists of a circular plate with the rim on both the sides divided into sixty units to measure time in *ghaṭikās*. The rim is also marked with signs of the zodiac in unequal segments according to their oblique ascensions, i.e., their periods of rising at the place of observation (*nijodaya*). The signs are marked in reverse order, i.e. proceeding counter-clockwise. Each of the signs is divided into two *horas* of 15°, three *dreṣkāṇas* or decans of 10°, into *navāṃśas* or ninths of 3°20' each, into twelfths of 2°10' each, and into *triṃśāṃśas* or thirtieths of 1° each. This type of six-fold division is called *ṣaḍvarga*. The polar axis passes through the centre of the plate so that the plate rests in a plane parallel to the celestial equator. Bhāskara teaches how to determine the ascendant (*lagna*) and time in *ghaṭikās* with this device.

Rāmacandra Vājapeyin includes the *Nāḍīvalaya-yantra* in his *Yantraprakāśa* of 1428. However, his instrument consisting of two circular plates appears to be somewhat different [Ms-YaPr, ch. 6.31–36].

5.3 Sarvatobhadra-yantra

While explaining how to mark the rim of the *Nāḍīvalaya* in six different ways (*ṣaḍvarga*) in his *Vāsanābhāṣya*, Bhāskara cites one and half verses in *Bhu-*

jaṅgaprayātā metre from his description of the *Sarvatobhadra-yantra*.[16] It appears from this citation that the *Sarvatobhadra-yantra* also consists of a circular plate with the rim marked in just the same way as the rim of the *Nāḍīvalaya-yantra* is calibrated. Since it bears a different name, it ought to differ from the *Nāḍīvalaya* in some manner, but it is not known in which manner. Nor is it known whether Bhāskara composed an independent manual on this instrument or whether he described this instrument in some unknown work. Unfortunately, this *Sarvatobhadra-yantra* is completely lost except for the citation mentioned above.[17]

5.4 Nāḍīvalaya at Jai Singh's observatory

When Sawai Jai Singh set up observatories with new types of masonry instruments in the early eighteenth century, he adopted Bhāskara's *Nāḍīvalaya* and caused it to be made in stone and plaster. But here the rim is marked only with degrees of arc and divisions of *ghaṭikās*, but not with zodiacal signs. Consequently, this instrument can be used just to measure time and not to determine the ascendant. This *Nāḍīvalaya* is set up at Jaipur and Banaras. In the Jaipur Observatory, the dials representing the north and south faces (to measure time when the sun is in the northern hemisphere and the southern hemisphere respectively) are separate. At Banaras the two dials are engraved on the two sides of the same stone disc. Therefore, this specimen is closer to Bhāskara's design (see Figure 8).

[16] [SiŚi1981, *Yantrādhyāya* vv. 5cd–7, *Vāsanābhāṣya*, p. 441].

[17] Nṛsiṃha Daivajña, at the commencement of his *Vāsanāvārttika* commentary on the *Siddhāntaśiromaṇi* [SiŚi1981, p. 5], refers to Bhāskara as *vasiṣṭhatulya-brahmatulya-sarvatobhadrādi-yantrāṇaṃ nirmātā*, implying that Bhāskara composed three texts on the instruments called *Vasiṣṭha-tulya-yantra*, *Brahma-tulya-yantra* and *Sarvatobhadra-yantra*, which is clearly a misreading. It should read *vasiṣṭhatulya-brahmatulya-sarvatobhadrayantrādīnāṃ kartā*; it would then mean that Bhāskara wrote *Vasiṣṭhatulya, Brahmatulya* and *Sarvatobhadra-yantra* and others. Of these *Brahmatulya* refers to the *Karaṇakutūhala*. In fact, in the very first verse of the *Karaṇakutūhala*, Bhāskara refers to this work as *Brahmasiddhāntatulya*; cf. [KaKu1991]. The other texts have not come down to us, nor were they cited by any other scholars.

Figure 8: *Nāḍīvalaya-yantra*, Jai Singh's Observatory at Banaras; From Sir Joseph Dalton Hooker, [Hoo1854, p. 68].

6 *Phalaka-yantra*

6.1 The Phalaka-yantra

The *Phalaka-yantra* [SiŚi1981, *Yantrādhyāya* vv. 16–27] is Bhāskara's own invention; he introduces it in these words [SiŚi1981, *Yantrādhyāya* v. 16]:

> दृङ्मण्डलेऽत्र स्फुटकाल उक्तः सुखेन नान्यैर्यतितं मयातः ।
> सद्गोलयुक्तेर्गणितस्य सारं स्पष्टं प्रवक्ष्ये फलकाख्ययन्त्रम् ॥
>
> *dr̥ṅmaṇḍale'tra sphuṭakāla uktaḥ sukhena nānyairyatitaṃ mayātaḥ |*
> *sadgolayukterganitasya sāraṃ spaṣṭaṃ pravakṣye phalakākhyayantram ||*
>
> Because others have not attempted to ascertain the apparent time (*sphuṭa-kāla*) easily from the vertical circle passing through the zenith and the planet (*dr̥ṅ-maṇḍala*), I have done so. I shall now explain lucidly the instrument named *Phalaka-yantra* which represents the essence of all computations (*gaṇitasya sāra*) pertaining to the true principles of Spherics (*sadgola-yukti*).

Because it is his own invention, he proudly presents the instrument along with a *maṅgalācaraṇa*, laying emphasis on the similarity between his name and that of the sun (*natvā … śrībhāskaraṃ bhāskaraḥ*). This verse contains two levels of meaning, one applied to the sun god and the other to the instrument.[18]

6.2 Construction and use of the Phalaka-yantra

The *Phalaka-yantra* consists of a rectangular board measuring 90×180 units on which 90 equidistant horizontal parallels are drawn. At the middle of the 30th parallel is inserted a pin to which an index is pivoted. With the pin as the centre, a circle is drawn with a radius of 30 units and the circle is divided in 60 *ghaṭikās* and 360 degrees. The instrument is used first to measure the altitude of the sun by direct observation and then to ascertain graphically the corrections for ascensional difference in order to determine time. It is a forerunner of the sine quadrant which would be introduced into India in the next centuries along with the astrolabe from the Islamic world. About this instrument Rai remarks as follows:

> The *Phalaka-yantra* described by Bhāskara II, though similar to the *Cakra-yantra* of Lalla and Śrīpati, is a great improvement on them as he uses in its construction the principles of spherical trigonometry so that it gives time, by the observation of the altitude of the Sun, much more accurately than that given by the instruments of earlier authors, though, once its principles are understood, the determination of time is very easy [Rai1985, pp. 328–332].[19]

6.3 Phalaka-yantra after Bhāskara

In the subsequent centuries, the *Phalaka-yantra* is included in some texts dealing exclusively with instruments. Thus Rāmacandra Vājapeyin includes it in his *Yantraprakāśa* of 1428 and describes its construction and use as stated by Bhāskara [Ms-YaPr, ch. 6.1–8]. In 1791, Nandarāma Miśra describes a *Phalaka-yantra* in his compendium *Yantrasāra*. However, this *Phalaka-yantra*

[18] Bhāskara, [SiŚi1981, *Yantrādhyāya* v. 17]. However, unlike in the other charming verses of this nature composed by Bhāskara, here the *śleṣa* is somewhat belaboured. Nityānanda, in his commentary *Marīci*, p. 379, draws out a third level of meaning by interpreting all the attributes as applicable to the author Bhāskara.

[19] See also [SL1861, pp. 214–218] and [Oha1994, pp. 286–290].

resembles more closely Bhāskara's *Nāḍīvalaya* [Ms-YaSā, f. 2r–v]. Moreover, during the survey of extant Indian astronomical instruments, I came across reports about five actual specimens of the *Phalaka-yantra*.

In the archives of the Museum of the History of Science at Oxford, there are photos of the front and back of an interesting astrolabe. An inscription on the front side of the suspension bracket informs that a certain Gaṅgāsahāya caused this astrolabe to be made by the artisan Rāmapratāpa at Tonk in Rajasthan in 1795. On the back of the *kursī* is an inscription which reads *cakra-turīya-phalaka-yantrāṇām ekatra samāveśaḥ*, '[Here is] the assemblage at one place the *Cakra-yantra*, the *Turīya-yantra* and the *Phalaka-yantra*' (see Figure 9). A *Turīya-yantra* (i.e. a quadrant) is normally incorporated on the back of the astrolabe for measuring the altitudes of the heavenly bodies. The back of the astrolabe can also function as the *Cakra-yantra*, which is a graduated circle with an upright peg at the centre. For using it as a *Phalaka-yantra*, starting a little above the centre, about 45 equidistant horizontal parallels are drawn up to the lower periphery on this astrolabe. Two concentric circles with different radii are drawn from the centre; these could be the equatorial circle and the Tropic of Cancer. The alidade of the astrolabe can, of course, function like the index required for the *Phalaka-yantra*, and can also measure the altitude of the sun. However, there is no circle divided into 60 *ghaṭikās* and 360 degrees of arc.

There are two more inscriptions providing the necessary parameters for using the *Phalaka-yantra*. An inscription on the upper left quadrant reads *ṭoṃkanagare phalaka-yantra-yogya | paramā yaṣṭiḥ* 36 | 34, 'In the city of Tonk, the maximum *yaṣṭi* appropriate for the *Phalaka-yantra* is 36 | 34.' The upper right quadrant bears the inscription *phalaka-yantra carajyā-khaṇḍāḥ* 3 | 2| 2 6 | 18 | 2, 'the segments of the sines of accensional differences for the *Phalaka-yantra* are 3 | 2 | 2 and 6 | 18 | 2.' Therefore Gaṅgāsahāya's attempt of using the back of the astrolabe as a *Phalaka-yantra* is historically very valuable. Unfortunately, the present location of the actual astrolabe is not known.

Sankara Balakrishna Dikshit reports that Sakhārāma Jośī, a resident of Koḍolī near Kolhapur, made several instruments including a *Phalaka-yantra* between 1790 and 1797 [Dik1981, p. 233n].

The third specimen was exhibited at the Lahore exhibition of 1864. It was described as 'Falk yantra —A graduated oblong plate, with inscribed circle and bar to give the shadow—for measurement of hours, angular distances, &c.' [Pow1872, p. 261]. Its present location is unknown.

A fourth specimen was caused to be made by the Maharaja of Benares and was presented to the Prince of Wales. When the Prince visited the Maharaja on 5 January 1876, the Maharaja presented him a set of tradi-

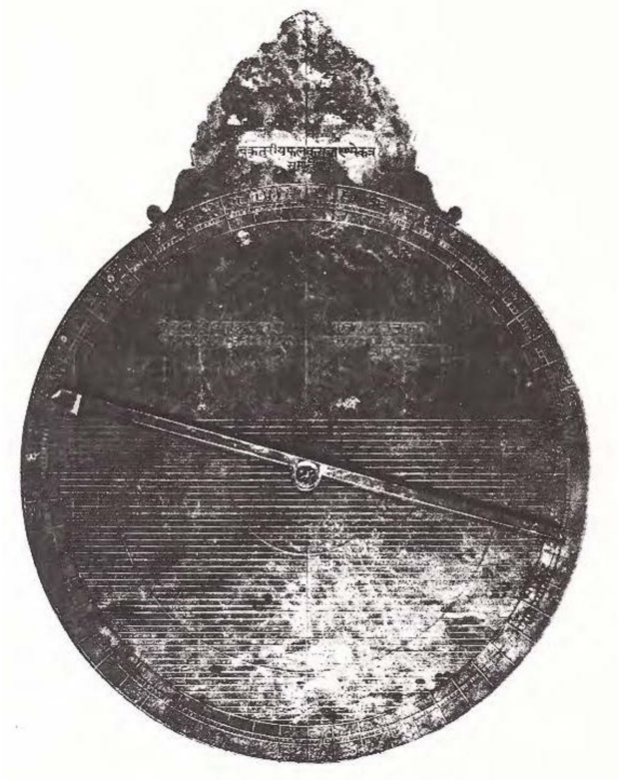

Figure 9: *Phalaka-yantra* engraved on the back of an astrolabe.

tional Indian astronomical instruments. There survives a list meticulously prepared by Bapu Deva Sastri. The list includes, aside from small replicas of the masonry instruments erected by Sawai Jai Singh in his observatories, the following instruments made according to the descriptions given by Bhāskarācārya: *Gola-yantra, Cakra-yantra, Cāpa-yantra, Turīya-yantra* and *Phalaka-yantra*. The list expressly mentions that the *Phalaka-yantra* was invented by Bhāskarācārya and that the details about all these instruments can be found in the *Siddhāntaśiromaṇi* and refers to the Bibliotheca Indica edition by 'Lancelot Wilkinson, Esq.' It is not known what happened to these instruments. These are not with any public museum in Britain; probably they were deposited in some royal palace and forgotten [Sar2014, pp. 12–15].

The fifth specimen of the *Phalaka-yantra* is with a dealer of scientific instruments in the US. It is a compendium of several astronomical instruments incorporated in a wooden casket, on which the dials of the instruments are painted very crudely by a certain Joshi Ramachandra in 1885 at Kuchaman

in Rajasthan. The casket can be used as a diptych sundial when the lid is lifted up. Such diptych sundials were very common in Europe. On the upper and lower sides, Ramachandra drew the dials of two traditional Indian instruments, viz., on the upper side the *Dhruvabhrama-yantra* invented by Padmanābha in about 1423 and on the other lower side the *Phalaka-yantra* invented by Bhāskarācārya in about 1150 (see Figure 10). In the *Phalaka-yantra*, there are several horizontal and vertical parallels. Upon this grid is drawn a circle which is divided into 60 *ghaṭikās* and numbered from 1–60, starting from the top and proceeding clockwise. Each *ghaṭikā* is subdivided into 3 units which correspond to 2°. At the centre is pivoted an index.

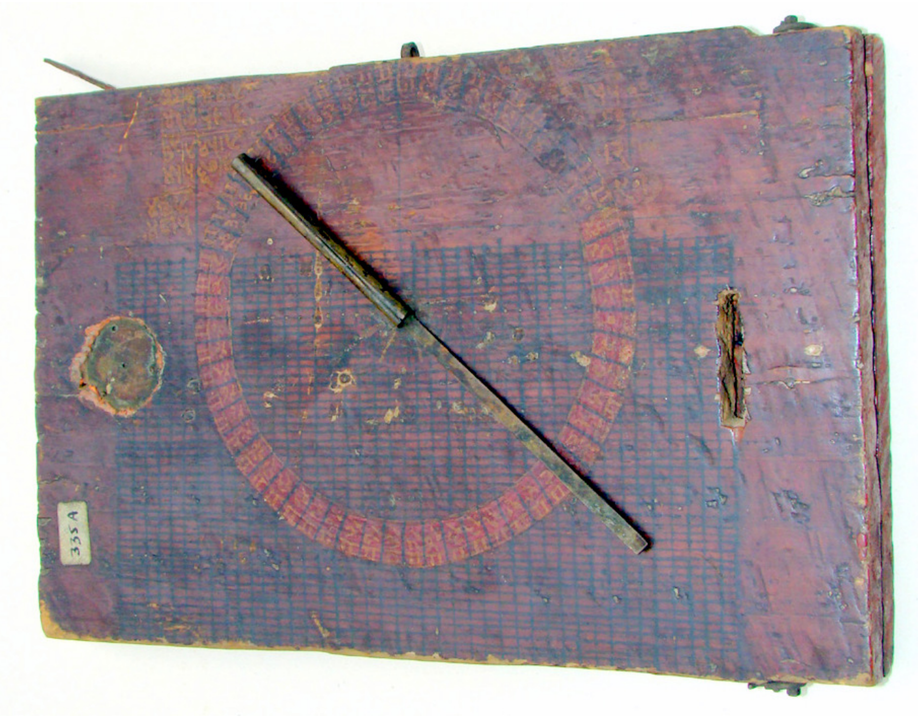

Figure 10: *Phalaka-yantra* painted on the bottom of a casket. (Photo courtesy of Dr. David Coffeen).

6.4 Svayaṃvha-yantras (Perpetual motion wheels)

Now we come to a class of instruments called *Svayaṃ-vaha-yantras* or 'self-propelled' instruments. In these instruments, a crucial role is played by mercury (*pārada, rasa*) to which many extraordinary properties are attributed in Indian imagination [SY1995, pp. 149–162]. After describing the proper astronomical instruments, Brahmagupta goes on to deal with some automatic devices based on the outflow water clock which consists of a graduated hollow cylinder with an aperture at the bottom [BSS1902, ch. 22.46–56]. On the surface of the water in this hollow cylinder is set up a float made of the outer shell of a dry gourd filled with mercury. To this float is tied a strip of cloth, in which 60 knots are made at regular intervals. As the water flows out of the aperture at the bottom and the level of the water in the cylinder goes down, the float also goes down pulling thereby the strip of cloth downwards. If the strip is attached skilfully to some objects, they can be made to perform certain tasks periodically. It is imagined that as the cylinder empties itself in sixty *ghaṭīs*, these objects can display the passage of each *ghaṭikā* with amusing periodic movements. Thus in a device named *Vadhūvara-yantra*, a sweetmeat in the form of a knot passes from the bridegroom's mouth to the bride's mouth at the passage of each *ghaṭikā*.[20] Lalla and Śrīpati accept these automata without any hesitation [ŚiDh1981b, ch. 21.10–17]; [SiŚe1932, ch. 9.7–11]. Bhāskara rejects them as rustic contrivances (*grāmya*) because they are dependent (*sāpekṣa*) on human agency to refill them every day with water. He prefers machines that turn on their own in an ingenious manner [SiŚi1981, *Yantrādhyāya* v. 57].

> यदथोरन्ध्रनलं तत् सापेक्षत्वात् स्वयंवहं ग्राम्यम् ।
> चतुरचमत्कृतकरी युक्तिर्यन्त्रं न हि ग्राम्यम् ॥
>
> *yad athorandhranalaṃ tat sāpekṣatvāt svayaṃvahaṃ grāmyam |*
> *caturacamatkṛtakarī yuktir yantraṃ na hi grāmyam ||*
>
> That self-propelled machine which consists of a cylinder with an aperture at the base is a rustic device (*grāmya*), because it is dependant (*sāpekṣa*). However, [the real self-propelled machine] is not the rustic piece, but one [that is based on a] design (*yukti*) which can astonish experts (*catura-camatkṛti-karī*).

6.5 Brahmagupta's perpetual motion wheel

In fact, Brahmagupta describes such a perpetual motion wheel in these words [BSS1902, ch. 22.53–54]:

[20] For details, see [Sar2008a].

लघुदारुमयं चक्रं समसुषिरारान्तरं पृथगाराणाम् ।
अर्धे रसेन पूर्णे परिधौ संश्लिष्टकृतसन्धिः ॥
तिर्यक्कीलो मध्ये द्व्याधारस्थोऽस्य परदो भ्रमति ।
छिद्राण्यूर्ध्वमधोऽतश्चक्रमजस्रं स्वयं भ्रमति ॥

laghudārumayaṃ cakraṃ samasuṣirārāntaraṃ pṛthag ārāṇām |
ardhe rasena pūrṇe paridhau saṃśliṣṭakṛtasandhiḥ ||
tiryakkīlo madhye dvyādhārastho'sya parado bhramati |
chidrāṇyūrdhvam adho'taścakram ajasraṃ svayaṃ bhramati ||

In [the rim of] a wheel made of light timber are inserted hollow spokes of equal size at equal intervals. [Each spoke is] half filled with mercury and the joints at the rim are sealed. In the centre of the wheel is inserted transversely an axis (*tiryak-kīla*), which when set up on two supports, the mercury runs up and down the hollow space [in the spokes], and the wheel turns perpetually (*ajasra*) on its own.

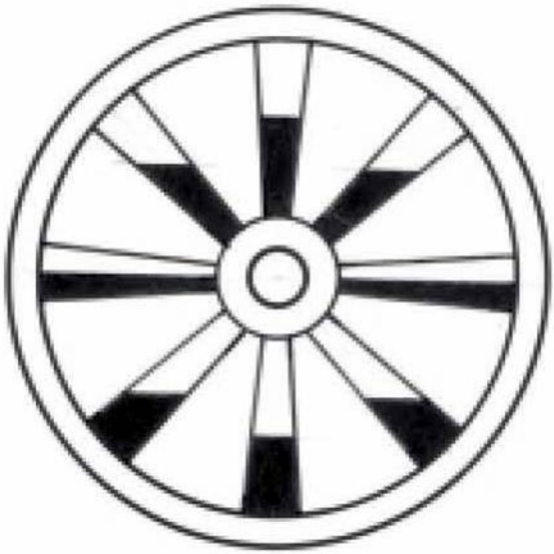

Figure 11: Brahmagupta's perpetual motion wheel.

6.6 Bhāskara's three models

Brahmagupta's idea of the perpetual motion machine was taken up enthusiastically and elaborated upon by Bhāskara who discusses three new models.

The *Vasanābhāṣya* describes their construction more fully and lucidly than the verses. Therefore, the relevant passages from the *Vāsanābhāṣya* are translated below.

Bhāskara's first model is like Brahmagupta's wheel, but the spokes are all slightly curved in one direction. This is an improvement on Brahmagupta's model because curved spokes offer less resistance and thus help the rotation of the wheel. Bhāskara describes his first model thus [SiŚi1981, *Yantrādhyāya* vv. 50–51*ab, Vāsanābhāṣya*, p. 476]:

> ग्रन्थिकीलरहिते लघुदारुमये भ्रमसिद्धे चक्र आराः ... समप्रमाणाः समसुषिराः समतौल्याः समान्तरा नेम्यां योज्याः। ताश्च नन्द्यावर्तवद् एकत एव सर्वाः किञ्चिद्वक्रा योज्याः। ततः तासाम् आराणां सुषिरेषु पारदस्तथा क्षेप्यो यथा सुषिरार्धमेव पूर्णं भवति। ततो मुद्रिताराग्रं तच्चक्रम् अयस्कारशाणवद्व्याधारस्थं स्वयं भ्रमति। अत्र युक्तिः। यन्त्रैकभागे रसो ह्यारामूलं प्रविशति। अन्यभागे त्वाराग्रं धावति। तेनाकृष्टं तत्स्वयं भ्रमति।
>
> *granthikīlarahite laghudārumaye bhramasiddhe cakra ārāḥ ... samapramāṇāḥ samasuṣirāḥ samataulyāḥ samāntarā nemyāṃ yojyāḥ | tāśca nandyāvartavad ekata eva sarvāḥ kiñcid vakrā yojyāḥ | tataḥ tāsām ārāṇāṃ suṣireṣu pāradas tathā kṣepyo yathā suṣirārdham eva pūrṇaṃ bhavati | tato mudritārāgraṃ taccakram ayaskāraśāṇavad dvyādhārasthaṃ svayaṃ bhramati | atra yuktiḥ | yantraikabhāge raso hyārāmūlaṃ praviśati | anyabhāge tv ārāgraṃ dhāvati | tenākṛṣṭaṃ tat svayaṃ bhramati |*
>
> Into the rim (*nemi*) of a wheel, which is obtained by means of a lathe (*bhramasiddha*) out of light timber, timber which is devoid of knots or thorns, should be inserted at equal intervals hollow spokes (*ārā*) of the same size, having the same hollow space inside (*suṣira*) and same weight. All the spokes should be slightly curved in the same direction like [the petals of] the *nandyāvarta* (Tabernaemontana coronaria) flower. Then into the hollow spaces of the spokes, mercury should be poured so that only half the hollow space [in each spoke] is filled. The ends of the spokes should then be closed and [the axis of] the wheel be set up on two upright supports like the ironsmith's whetstone (*ayaskāra-śāṇa*). Then the wheel turns by itself. The reason (*yukti*) for this is as follows: [because of the curvature of the spokes] the mercury in one part of the wheel rushes towards the base of the spokes (*ārāmūla*) while in another part it rushes towards the top of the spokes (*ārāgra*). Impelled by this [internal movement], the wheel turns automatically [and will continue to do so].

Besides thus modifying Brahmagupta's wheel, Bhāskara proposes two other variants.

In the second model, a groove is cut in the rim of the wheel and filled half with water and half with mercury. The water, while trying to flow downwards, pushes the mercury and vice versa, this internal tension resulting in the rotation of the wheel itself.

Having excavated the entire length of the rim on a lathe (*bhrama-yantra*) to produce a groove (*suṣira*) having a depth (*vedha*) as well as a width (*vistāra*) of two *aṅgulas*, the top of the groove should be covered with Palmyra (*tāla*,

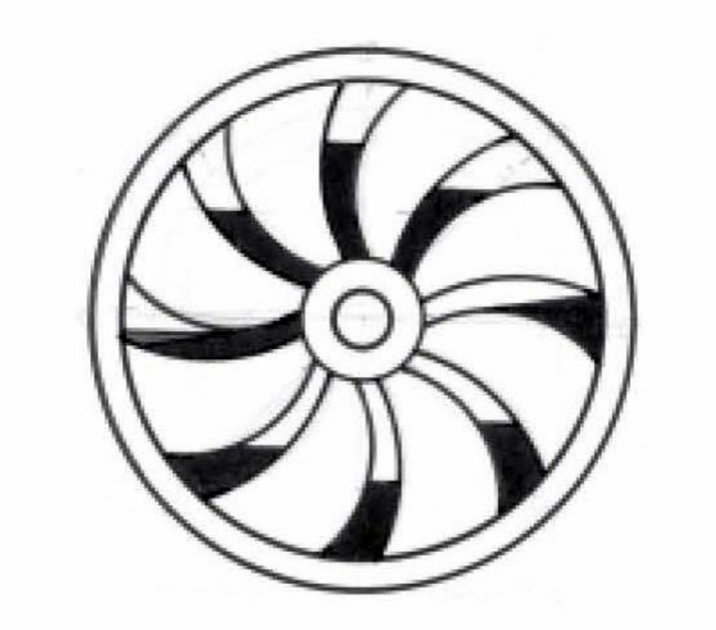

Figure 12: Bhāskara's first model with the spokes curved like the petals of the *Nandyāvarta* flower.

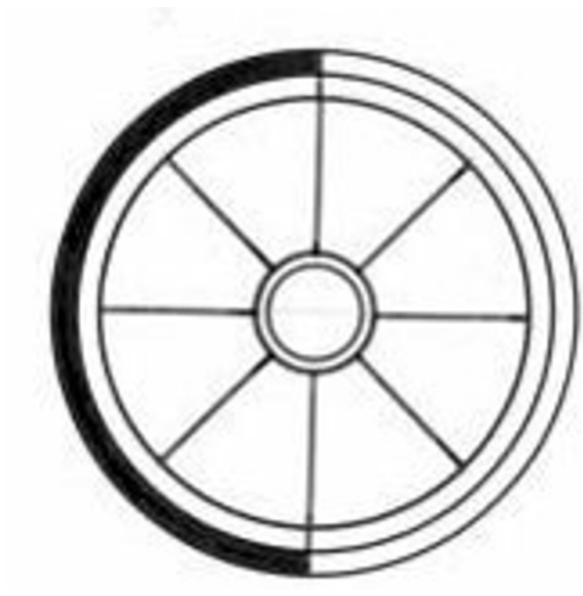

Figure 13: Bhāskara's second model.

Borassus flabelliformis) leaves or others and sealed with beeswax (*madana*) or other substance. The axis (*akṣa*) of this wheel also should be set up on two upright posts [as before]. Then at the top of the rim, the Palmyra leaf should be pierced and [and through this hole] mercury poured into the groove until the lower [half] of the groove is filled by mercury. Then water should be poured on one side (i.e., in the upper part of the groove). This water will not be able to push the mercury because, though it is liquid, mercury is [exceedingly] heavy. After the hole [in the palm leaf is sealed], the wheel, impelled by the water, turns on its own.[21]

Bhāskara's third model is much more complex; it is based on water wheels (*jala-yantra, jala-ghaṭī-yantra, araghaṭṭa*) employed in irrigation.[22] Here a series of pots are tied to the rim of a large wheel. As the wheel is turned by human or animal power, the pots scoop up water from a low-lying river or a pond and discharge it into irrigation channels which are at a higher level. Bhāskara suggests that these pots should be successively filled with water from a reservoir by means of a siphon. The pots at the top, when filled with water, become heavy and move downwards, causing the wheel to turn. The water discharged from the pots below is made to flow into the reservoir by a channel below; thus there will be no need to refill the reservoir again [SiŚi1981, *Yantrādhyāya* vv. 53*cd*–56, *Vāsanābhāṣya*, p. 477].

> अथ चक्रनेम्यां घटीर्बध्वा जलयन्त्रवद्व्याधाराक्षसंस्थितं तथा निवेशयेद्यथा नलकप्रच्युतजलं तस्य घटीमुखे पतति। एवं पूर्णघटीभिराकृष्टं तद्भ्रमत् केन निवार्यते। अथ चक्रच्युतस्योदकस्य अधःप्रणालिकया कुण्डगमने कृते पुनर्जलप्रक्षेपनिरापेक्ष्यम्॥
>
> *atha cakranemyāṃ ghaṭīrbadhvā jalayantravad dvyādhārākṣasaṃsthitaṃ tathā niveśayed yathā nalakapracyutajalaṃ tasya ghaṭīmukhe patati | evaṃ pūrṇaghaṭībhir ākṛṣṭaṃ tadbhramat kena nivāryate | atha cakracyutasyodakasya adhaḥpraṇālikayā kuṇḍagamane kṛte punarjalaprakṣepanairapekṣyam ||*
>
> Now tie up [a series of] pots along the rim of a wheel, set the axle (*akṣa*) of the wheel on two supports, as in the case of the water wheel (*jala-yantra*), in such a manner that the water discharged from the siphon (lit. tube, *nalaka*) falls into the pots. Pulled downwards by the pots filled with water, the wheel turns and

---

[21] यन्त्रनेमिं भ्रमयन्त्रेण समन्ताद् उत्कीर्य द्व्यङ्गुलमात्रं सुषिरस्य वेधो विस्तारश्च यथा भवति ततस्तस्य सुषिरस्योपरि तालपत्रादिकं मदनादिना संलग्नं कार्यम्। तदपि चक्रं द्व्याधाराक्षस्थितं कृत्वोपरि नेम्यां तालदलं विद्ध्वा सुषिरे रसस्तावत् क्षेप्यो यावत्सुषिरस्याधोभागो रसेन मुद्रितः। पुनरेकपार्श्वे जलं प्रक्षिपेत् तेन जलेन द्रवोऽपि रसो गुरुत्वात्परतः सारयितुं न शक्यते। अतो मुद्रितच्छिद्रं स्वयं भ्रमतीति।

yantranemiṃ bhramayantreṇa samantād utkīrya dvyaṅgulamātraṃ suṣirasya vedho vistāraśca yathā bhavati tatastasya suṣirasyopari tālapatrādikaṃ madanādinā saṃlagnaṃ kāryam | tadapi cakraṃ dvyādhārākṣasthitaṃ kṛtvopari nemyāṃ tāladalaṃ viddhvā suṣire rasas tāvat kṣepyo yāvat suṣirasyādhobhāgo rasena mudritaḥ | punarekapārśve jalaṃ prakṣipet | tena jalena dravo'pi raso gurutvāt parataḥ sārayituṃ na śakyate | ato mudritacchidraṃ svayaṃ bhramatīti |

[SiŚi1981, *Yantrādhyāya* vv. 50–51*ab, Vāsanābhāṣya*, p. 476]

[22] On the history of water lifting devices in India, see [Hab2012, pp. 8–14].

who can stop it! Now, when it is so arranged that the water discharged from the pots is transported to the reservoir (*kuṇḍa*) by means of a channel below (*adhaḥ-praṇālikā*), there will be no need to fill the reservoir again and again.

Bhāskara, however, does not explain how the channel below should be constructed and, more important, how the water discharged by the pots into the channel below is made to go into the overhead reservoir which is by definition at a level higher than the water wheel.[23]

6.7 Bhāskara on the siphon

Interestingly, at this place Bhāskara describes the siphon and its working principle in great detail as if it were a novelty. As we saw earlier in connection with the water clock, Bhāskara is usually indifferent towards mechanical details. But now he describes the siphon elaborately with an analogy from daily life. The history of the siphon and of the many similar tools and devices has yet to written. Since descriptions of these are very rare in Sanskrit, the full passage from the *Vāsanābhāṣya* is translated below [SiŚi1981, *Yantrādhyāya* vv. 53*cd*–56, *Vāsanābhāṣya*, p. 477]:

ताम्रादिधातुमयस्य अङ्कुशरूपस्य वक्रीकृतस्य नलस्य जलपूर्णस्य एकमग्रं जलभाण्डेऽन्यदग्रं बहिरधोमुखं चैकहेलया यदि विमुच्यते तदा भाण्डजलं सकलमपि नलेन बहिः क्षरति। तद्यथा। छिन्नकमलस्य कमलिनीनलस्य जलभृद्भाण्डे क्षिप्तस्य जलपूर्णसुषिरस्य एकमग्रं भाण्डाद्बहिरधो-मुखं द्रुतं यदि ध्रियते तदा भाण्डजलं सकलमपि नलेन बहिर्याति। इदं कुक्कुटनाडीयन्त्रम् इति शिल्पिनां हरमेखलिनां च प्रसिद्धम्। अनेन बहवश्चमत्काराः सिद्ध्यन्ति।

tāmrādidhātumayasya aṅkuśa-rūpasya vakrīkṛtasya nalasya jalapūrṇasya ekamagraṃ jalabhāṇḍe'nyadagraṃ bahiradhomukhaṃ caikahelayā yadi vimucyate tadā bhāṇḍajalaṃ sakalamapi nalena bahiḥ kṣarati | tadyathā | chinnakamalasya kamalinīnalasya jalabhṛdbhāṇḍe kṣiptasya jalapūrṇasuṣirasya ekamagraṃ bhāṇḍād bahiradhomukhaṃ drutaṃ yadi dhriyate tadā bhāṇḍajalaṃ sakalamapi nalena bahiryāti |

[23] In the Lahore Exhibition of 1864, in the section of the 'Mathematical and Philosophical Instruments,' an instrument was displayed which was inspired by Bhāskara's third model. It was described thus in [Pow1872, p. 261] '31. *Swayambahu yantra*—A revolving disc turning by a stream of water, which flows by a siphon tube out of a vessel properly placed and dropped on the cogs or teeth of the disc. A revolution is maintained at a certain rate, and by this means time is ascertained after the machine having been once started at a known hour.' It must have been constructed specially for the Exhibition according to Wilkinson's translation of the *Golādhyāya*. It is not known whether it is still extant in the stores of the Lahore Museum, which was established in the same building that was originally constructed for the Exhibition.

> *idaṃ kukkuṭanāḍī-yantram iti śilpināṃ haramekhalināṃ ca prasiddham | anena bahavaścamatkārāḥ siddhyanti |*
>
> Take a tube, made of copper or any other metal; bend it like the elephant's goad (*aṅkuśa*), fill it completely with water. Hold one end in water in a vessel and the other end is left outside so that it faces downwards. When both ends of the tube are opened at the same time, all the water in the vessel flows out through the tube and falls outside the vessel. This happens in this way. Take a lotus stalk (*kamalinī-nala*) and remove the lotus flower from it. The holes of the stalk are filled with water. Put [one end of] the stalk in a vessel filled with water and release the other end outside the vessel; all the water in the vessel flows out through the stalk. This [device] is well known to the artisans (*śiplins*) and to the *haramekhalins* under the name *kukkuṭa-nāḍī-yantra*.[24] With this [device] many amusing tricks (*camatkāra*) can be performed.

Obviously inspired by Bhāskara, Rāmacandra Vājapeyin describes very briefly in his *Yantraprakāśa* two instruments named *Kāhalā-yantra* and *Hasti-yantra* in which a siphon is incorporated [Ms-YaPr, ch. 6.59–60]. The former device is shaped like a drum that tapers on both the sides (*kāhalā*); a siphon built into this drum draws the water upwards from a river and discharges it outside. The second device is constructed in the form of an elephant (*hastin*) in which a siphon is incorporated in such a manner that the elephant drinks water from a river with its trunk and discharges it outside as urine. Even if we assume that these devices operate forever when once they are set up, what purpose do they serve? If it is merely to lift water from low-lying areas, the devices need not be disguised as a drum or an elephant. In the form of a drum or an elephant that constantly urinates, they would just serve to entertain people, say in weekly market or in an annual festival. Or, in Bhāskara's words, these are just *camatkāras*.

But what exactly is the purpose of including these devices in the *Yantrādhyāya*, which deals with instruments to measure time, altitudes and zenith distances and other such parameters? Bhāskara himself does not see any connection between the serious pursuit of astronomy and these self-propelling devices. He says that these have no relevance in elucidating the principles of Spherics (*nedaṃ golāśritayā [vidyayā]*). But then why did he take the trouble of discussing them in his book, even going to the extent of suggesting improvements on Brahmagupta's original model and proposing two new models? For our disappointment, the Ācārya says that he discussed these devices here only because the previous astronomers like Brahmagupta had discussed them (*pūr-*

[24] The terms *kukkuṭa-nāḍī-yantra* and *haramekhalin* do not occur anywhere else. Monier-Williams' dictionary explains that *haramekhalin* is 'a particular class of artisans', on the authority of [Bhāskara's] *Golādhyāya*! If so, the term *śilpin-s* would subsume *haramekhalin-s*; but Bhāskara treats them as separate entities by mentioning them separately. There is a medical text named *Haramekhalā* said to have been taught by Hara to Pārvatī.

voktatatvān mayāpy uktam).[25] But Bhāskara's discussion of perpetual motion machine served at least one purpose, that of generating a debate about the transmission of perpetual motion machines to Europe.[26]

6.8 Perpetual motion machines in Europe

A perpetual motion machine or *perpetuum mobile* is a device which is supposed to perform useful work without any external source of energy or, at least, where the output is far greater than the input [AD1986, p. 71]. The idea of constructing such machines and of employing the power generated by them for useful purposes has fascinated the minds of many inventors in Europe since the Middle Ages. Various attempts were made to construct such perpetual motion machines, but none succeeded in fashioning a perfect *perpetuum mobile*.[27] Modern science ridicules these attempts as mere flights of fantasy.

The American historian Lynn White, however, argues that such fantasies are also important in the history of ideas and that the concept of perpetual motion was a significant element in Europe's thinking about mechanical power. White traces the origin of the perpetual motion machine to Bhāskara of the twelfth century. He states that Bhāskara's two models of mercury filled wheels were immediately taken up by the Islamic world and amplified. The Islamic world in turn transmitted the idea to the West at the beginning of the thirteenth century, together with Indian numerals and the decimal place-value system.

Europe responded to this idea of *perpetuum mobile* with great enthusiasm, and the engineers there began to design several new models. In contrast to India and the Islamic world, the medieval engineers of Europe tried to apply the idea of perpetual motion for practical purposes, for the material benefit of mankind. Already the industrial application of water-power and wind-power was revolutionizing manufacture, and the two new forces introduced by the

[25] एवं बहुधा यन्त्रं स्वयंवहं कुहकविद्यया भवति ।
नेदं गोलाश्रितया पूर्वोक्ततत्वान्मयाप्युक्तम् ॥
evaṃ bahudhā yantraṃ svayaṃvahaṃ kuhakavidyayā bhavati |
nedaṃ golāśritayā pūrvoktatatvānmayāpyuktam || [SiŚi1981, *Yantrādhyāya* v. 58]

In this manner several perpetual motion machines can be constructed. But these belong to the juggler's art (*kuhaka-vidyā*), and not to the theory of Spherics. [Even so] I discussed these because they were described by ancients.

[26] For details of this debate and its resolution, see [Sar2008b, pp. 64–75].

[27] The history of perpetual motion machines and the accounts of many charlatans who pretended to build these are quite fascinating; the internet is full of interesting entries.

Islamic world, viz. gravity and magnetism, appeared to operate with a constancy unrivalled by wind and water. White concludes his thesis with these words:

> Thus the Indian idea of perpetual motion [...] not only helped European engineers to generalize their concept of mechanical power, but also provoked a process of thinking by analogy that profoundly influenced Western scientific views [Whi1978, pp. 43–57, esp. 56–57].

Lynn White's thesis that the foundations of modern power technology lay in the idea of perpetual motion was generally accepted by historians of technology,[28] but his attempt to trace the origin of perpetual motion machines to twelfth-century India was contested on chronological grounds. In their excellent work *Islamic Technology*, Ahmad Y. Al-Hassan and Donald R. Hill argue that such machines were known to the Arabs long before Bhāskara's time; they mention a manuscript which can be assigned roughly to a period between the ninth and twelfth centuries [AD1986, pp. 70–71]. Joseph Needham, on the other hand, asserts that both the Indian and Arabic accounts owe their inspiration to China, in particular to the clock tower designed by Su Sung in 1090 [Nee1965, vol. IV.2, p. 540].

The remarkable aspect of this debate is that all the parties based their argument on two English translations of Sanskrit texts available to them, namely Wilkinson's translation of the *Golādhyāya* of Bhāskara and Ebenezer Burgess' translation of the *Sūryasiddhānta*. What they did not know was that the idea of perpetual motion did not originate in the twelfth-century book of Bhāskara but in the seventh century text of Brahmagupta which predates the Arabic manuscript and Su Sung's clock tower. Moreover, Brahmagupta's work is known to have been transmitted to the Abbasid court whereas no such transmission of Bhāskara's works has taken place.

Therefore, if credit is to be given for conceiving the notion of a perpetual motion machine, it should go to Brahmagupta. But then the Arab writers who elaborated on Brahmagupta's notion or independently conceived of mercury wheels themselves, as also the European engineers who contributed to the modern power technology deserve credit to an equal degree.

[28] Cf. [Nee1965, vol. IV.2, p. 54]: 'Lynn White has done a good service by pointing out that in correct historical perspective, the idea of perpetual motion has heuristic value.'

7 Conclusion

Going back to the astronomical instruments proper as described by Bhāskara, one is struck by the utter simplicity of design in these instruments. Leaving aside the water clock, the rest of the instruments can be reduced to two basic types: a wooden circle with a graduated rim, and a wooden staff. With these two, the astronomer measured the zenith distance as an arc and the gnomonic shadow as a line, and calculated the rest from such simple measurements. These instruments are such that they can be manufactured everywhere with little or no skill. The basic idea seems to be that a tool or an instrument is just a means to an end. The end may be elaborate but the means must be the simplest possible.

However, Bhāskara lived at the turning point of the history of astronomical instrumentation in India. Soon these instruments described from Brahmagupta up to Bhāskara became obsolete with the introduction of the astrolabe which incorporates also the sine quadrant. Commenting on Bhāskara's *Cakra-yantra* which consisted of a circular plate with a graduated rim, Nṛsiṃha Daivajña remarks in his *Vāsanāvārttika* of 1621 that 'the same *Cakra-yantra* becomes astrolabe (*Yantrarāja*) when it is equipped with latitude plates (*akṣa-patra*) and rete (*bha-patra*)'[29] and then goes on to explain the construction of the various components of the astrolabe and their functions, citing extensively from Mahendra Sūri's *Yantrarāja* of 1370 and Rāmacandra Vājapeyin's *Yantraprakāśa* of 1428 [SiŚi1981, pp. 445–457].[30]

Nṛsiṃha wonders why the great Bhāskarācārya did not include the astrolabe (*yantrarāja*) in his repertoire of astronomical instruments and offers a somewhat tame explanation: 'Bhāskarācārya did not discuss the astrolabe because its construction varies according to the latitude (*deśa*) and time (*kāla*).'[31]

This is true, but only partially: in the astrolabe the latitude plates (*akṣa-patras*) are latitude-specific and have to be calibrated according to the observer's latitude; an important component called rete (*bha-patra*) has the positions of the several bright stars marked on it; these positions vary as the longitudes of the stars change due to precession. But what Nṛsiṃha fails to

[29] इदमेव चक्रयन्त्रमक्षपत्रभपत्रयुतं यन्त्रराज इत्याहुः ।
idameva cakrayantram akṣapatrabhapatrayutaṃ yantrarāja ity āhuḥ |

[SiŚi1981, p. 444]

[30] This long digression of Nṛsiṃha Daivajña with many citations is very valuable and deserves a separate treatment.

[31] यन्त्रराजाख्यमिदं यन्त्रं देशभेदात् कालभेदाच्च विसदृशनिर्माणम् इति भास्कराचार्यैर्नोक्तम्।
yantrarājākhyam idaṃ yantraṃ deśabhedāt kālabhedācca visadṛśanirmāṇam iti bhāskarācāryairnoktam |

[SiŚi1981, p. 457]

state is that the construction of the *akṣa-patra* and *bha-patra* is not as simple as that of the *Cakra-yantra*, for the construction of these components of the astrolabe is based on stereographic projection which was unknown to Bhāskara.

While commenting on Bhāskara's description of the quadrant (*Turyagola-yantra*), Nṛsiṃha discusses Padmanābha's *Dhruvabhrama-yantra* which incorporates the sine quadrant on its reverse side, with long extracts from Padmanābha's work. After discussing the advantages of the quadrant, Nṛsiṃha again wonders why the great Bhāskarācārya did not discuss the uses of the quadrant and answers by saying that Bhāskara did not do so because he did not wish to extend the length of his book.[32]

But the real reason for Bhāskara not discussing the astrolabe and the sine quadrant was that these were not known in India well enough in the twelfth century. It is only in the second half of the fourteenth century that Mahendra Sūri composed the first ever Sanskrit manual on the astrolabe, where he gave the Sanskrit name *yantrarāja* to the astrolabe [Sar2008d, pp. 240–256]. And the sine quadrant was described in Sanskrit for the first time by Padmanābha around 1423 [Sar2012, pp. 321–343]. After the introduction of these two instruments, the older instruments described by astronomers from Brahmagupta to Bhāskara became obsolete. Nonetheless, just as the texts on these older instruments continued to be studied as part of the study of *Jyotiḥśāstra*, so too these instruments also must have been constructed and used until recent times, along with the astrolabe and sine quadrant with Sanskrit legends.

[32] *tadvistarabhayānnehoktam* [SiŚi1981, p. 459].

Part VI

The Karaṇakutūhala

गणेशं गिरं पद्मजन्माच्युतेशान्
ग्रहान् भास्करो भास्करादींश्च नत्वा ।
लघुप्रक्रियां प्रस्फुटं खेटकर्म
प्रवक्ष्याम्यहं ब्रह्मसिद्धान्ततुल्यम् ॥

gaṇeśaṃ giraṃ padmajanmācyuteśān
grahān bhāskaro bhāskarādīṃś ca natvā |
laghuprakriyāṃ prasphuṭaṃ kheṭakarma
pravakṣyāmy ahaṃ brahmasiddhāntatulyam ||

I, Bhāskara, having worshipped Lord Gaṇeśa, [the Goddess of] Speech, Brahmā, Viṣṇu, Maheśvara, and all the planets commencing from the Sun, venture to clearly set forth simple procedures for planetary computations which [give results that] are compatible with the *Brahmasiddhānta*.

Bhojarājā and Bhāskara: Precursors of *Karaṇakutūhala* algebraic approximation formulas in the *Rājamṛgāṅka*

Kim Plofker[*]

1 Introduction

Telling time by the length of gnomon shadows is a very ancient practice. Prior to the development of spherical cosmological models and accompanying trigonometric techniques in the last few centuries BCE, such computations relied on non-trigonometric algorithms approximately relating shadow lengths to time and geographic location. Even after the full implementation of trigonometric techniques, various algebraic approximations were still employed in the "Three Questions" (*tripraśna*) problems of Sanskrit mathematical astronomy concerning the determination of cardinal directions, geographic position, and time for the observer's locality.

The illustrious Bhāskarācārya prescribed several such approximations in his *karaṇa* or astronomical handbook *Karaṇakutūhala* (epoch date 1183 CE). Some of these have recently been explored in [Plo2016], especially in relation to the famous algebraic approximation to the sine function associated with Bhāskara I [Gup1986], [Hay1991], [Shi2011]. The present paper examines some of their predecessors earlier in the *karaṇa* tradition.

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1.1 Shadow rules from ancient texts

Some of the earliest surviving Sanskrit mathematical astronomy literature preserves pre-trigonometric rules relating length of a gnomon shadow to time of day, which continued to appear in medieval treatises long after trigonometrically exact versions were known. Hayashi [Hay2017, pp. 7–8] describes versions of such a formula attested in sources ranging from the late first-millennium BCE *Arthaśāstra* to astronomical *siddhāntas* of the tenth and eleventh century. All the versions are equivalent to the following expression:

$$t = \frac{g}{2(s+g)}\, d \tag{1}$$

where d is the length of daylight in *ghaṭikās* at the given latitude and time of year, t is the elapsed time since sunrise in the morning (or the remaining time till sunset in the afternoon), g is the length of the vertical gnomon in the linear units *aṅgulas* or digits, and s is the length of the shadow in digits at the given time.

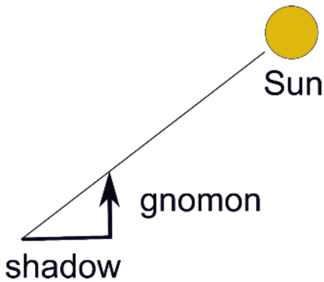

Figure 1: Right triangle formed by a vertical gnomon, a sun ray, and its shadow.

Hayashi points out that the presentation of the rule "suggests that the origin of these formulas lies in the idea that the time (t) is inversely proportionate to the sum of the lengths of the gnomon (g) and the shadow (s) which constitute the two orthogonal sides of the right triangle produced by the eclipse of the sun ray" (see Figure 1). If the noon shadow s_n is approximately zero to make the elapsed time at noon equal to half the day, and the shadow at sunrise or sunset is considered infinitely long, the relation serves quite well as a rule of thumb for telling time.

1.2 Trigonometric shadow rules

With the advent of plane-trigonometric methods to compute arcs of reference circles on the terrestrial and celestial spheres, the problem of telling time from gnomon shadows in Sanskrit texts became primarily an elaborate exercise in right-triangle geometry. Dozens of mathematically equivalent similar-triangle proportion rules employing many different line segments and arcs were devised in Sanskrit treatises; see the *siddhānta* of Vaṭeśvara [VaSi1986, ch. 1.178–196] for perhaps the most comprehensive set of examples. Here we will limit ourselves to defining standard concepts and notation for these calculations.

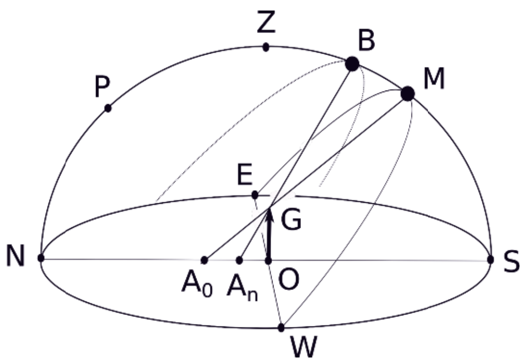

Figure 2: The geometry of noon shadows of a vertical gnomon. Meridian positions of the sun are shown at M on the celestial equator (on the equinox) and at B north of the equator.

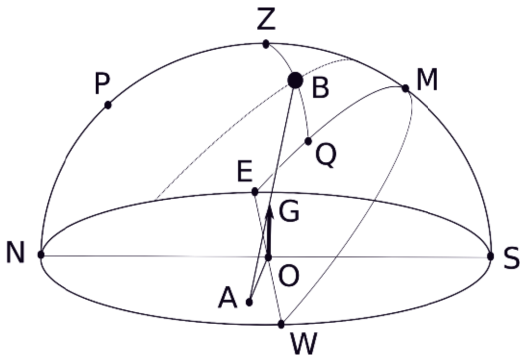

Figure 3: The shadow of a gnomon when the sun at B north of the celestial equator has a non-zero *nata*, i.e., is not on the meridian.

Figures 2 and 3 show the visible half of the celestial sphere above the local horizon $NESW$, centered at O with zenith Z, celestial north pole P, local meridian $NPZMS$, and celestial equator EMW. In Figure 2, rays from the noon sun on the local meridian extend between the tip G of the 12-digit vertical gnomon OG and the tip of the noon shadow, falling at A_0 when the sun is at M on the celestial equator and A_n when it is at B on a northern day-circle parallel to the equator. The (northern) local terrestrial latitude ϕ is the arc ZM of the meridian, while the (northern) ecliptic declination δ of the sun at B is the arc BM. The noon equinoctial shadow $s_0 = A_0O$ and its hypotenuse $h_0 = A_0G$ are given by the relations

$$s_0 = 12\,\frac{R\sin\phi}{R\cos\phi}, \qquad h_0 = \frac{12R}{R\cos\phi} = \sqrt{{s_0}^2 + 12^2}, \tag{2}$$

where the scaled trigonometric functions R sin and R cos are equivalent to the modern sine and cosine times a non-unity trigonometric radius R.

When the sun has a nonzero declination δ, it also has a nonzero "half-equation of daylight" ω, defined by the expression

$$R\sin\omega = R\cdot\frac{R\sin\delta}{R\cos\delta}\cdot\frac{R\sin\phi}{R\cos\phi}. \tag{3}$$

The length of a given day is equal to the length of the equinoctial day or 30 *ghaṭikās* plus two times ω, likewise expressed in *ghaṭikās*. The noon shadow $s_n = A_nO$ on that day and its hypotenuse $h_n = A_nG$ are given by

$$s_n = 12\,\frac{R\sin(\phi-\delta)}{R\cos(\phi-\delta)}, \qquad h_n = \frac{12R}{R\cos(\phi-\delta)} = \sqrt{{s_n}^2 + 12^2}. \tag{4}$$

If the sun is south of the equator this arc becomes $\phi + \delta$ instead.

When the sun is not on the local meridian, as in Figure 3, the tip of the gnomon shadow falls at A, away from the north-south line. The sun has a nonzero *nata* n or arc MQ of the equator corresponding to the arc of its parallel day-circle between its current position and the meridian. The complement of n extending to the east or west point of the horizon is the *unnata*. The lengths of the shadow s and its hypotenuse h depend on ϕ, δ and n as follows (continuing to assume for convenience that ϕ and δ are both northern):

$$s = \sqrt{h^2 - 12^2}, \qquad h = \frac{R + R\sin\omega}{R + R\sin\omega - (R - R\cos(n))}\cdot\frac{12R}{R\cos(\phi-\delta)}. \tag{5}$$

As before, if the declination is southern $\phi - \delta$ becomes $\phi + \delta$ and $R + R\sin\omega$ becomes $R - R\sin\omega$ in this expression.

2 Shadow approximations in Bhojarāja and Bhāskarācārya

Table 1: Numbers of the verses covering various topics in the *tripraśna* chapters of the *Rājamṛgāṅka* [RaBh1987, pp. 12–16] and the *Karaṇakutūhala* [KaKu1991, pp. 36–47].

Topic	*Rājamṛgāṅka*	*Karaṇakutūhala*
Ascensional differences	1–2	1–2
Ascendant	3–9	2–4
Time from ascendant	9–11	5–6
Length of day and night		7
nata and *unnata*	12–14	7
Longest day and noon shadow	15–16	
Current time from shadow	17–18	
Shadow/hypotenuse from *nata*	19–22	8–10
***nata* from hypotenuse**	23–25	11–12
R sin of latitude and altitude	26	
Arcs from R sines by table interpolation	27–29	
Planetary latitudes, velocities, tropical longitudes	30–44	
List of declination-differences		13
Finding declination by interpolation	45–46	13–14
Approximation for declination		15
Approximation for latitude		16
Planetary true declination	47–48	
Planetary "day" length, time, etc	49–57	

The first known Sanskrit astronomical handbook of the second millennium is the *Rājamṛgāṅka* with epoch date Śaka 964 = 1042 CE attributed to Bhojarāja (r. ca. 1005–1055), a Paramāra king of Dhārā; see [Rag2006], [CESS, A4.336–339], and [RaBh1987]. In its surviving form it is a rather sprawling work in several hundred verses, and appears to be a sort of hybrid of the *karaṇa* and *sāraṇi* or table-text genres [MP].

The *Rājamṛgāṅka*'s better-known successor, the Śaka 1105 = 1183 CE *Karaṇakutūhala* of Bhāskarācārya, is far more concise and elegantly constructed in a traditional verse-only format, with no appended numeric-array tables or instructions for using them. The distribution of verses on computing direction, place and time in the *tripraśna* chapters of the two works is shown in Table 1, where the particular rules we will focus on in the remainder of this paper are bolded.

Despite their differences, the two handbooks follow roughly the same order in the shared subject matter of their *tripraśna* chapters. The excerpts in Table 2 illustrate their different expository approaches on the subjects of seasonal variation in day length and the *nata* and *unnata*. The *Rājamṛgāṅka* takes three rather prolix *anuṣṭubh* verses to express what the *Karaṇakutūhala* mostly summarizes in one neat *mālinī* stanza.

But a more interesting feature of these *tripraśna* chapters appears in the subsequent sets of *anuṣṭubh* and *drutavilambita* verses shown in Table 3. In these verses, Bhojarāja and Bhāskara present approximation rules for finding the gnomon shadow and hypotenuse at a desired time "without sines": that is, by algebraic formulas depending only on the given day-length and the local noon or noon equinoctial shadow. Both texts here prescribe algorithms commencing with the length of the given day decreased by ten (or its half decreased by five). The *Rājamṛgāṅka* then proceeds to define a multiplier M dependent on ω and n and a divisor $M-1$, which are combined with the noon shadow s_n to give an approximation for the shadow s at the desired time equivalent to the following:

$$M = \frac{9(20+2\omega^{(\mathrm{gh})})}{n^2} + \frac{20+2\omega^{(\mathrm{gh})}}{100}, \quad s \approx \frac{\sqrt{(M+M-1)\cdot(12)^2+(s_n\cdot M)^2}}{M-1} \tag{6}$$

The *Rājamṛgāṅka*'s "day-length minus ten" is here represented as $20+2\omega$ *ghaṭikās*. Note that this formula for s does not apply to the special case of noon: the appropriate noon shadow-length s_n is assumed to be already known, and obtaining the multiplier M for it would require dividing by $n=0$.

The *Karaṇakutūhala*'s stanzas, by contrast, construct somewhat simpler formulas for the shadow-hypotenuse h rather than the shadow itself, including the special case of the noon shadow h_n:

[1] Edition has *natād athonnatāt*.

Table 2: Verses on day length and *nata/unnata* in the *Rājamṛgāṅka* [RaBh1987, p. 12] and *Karaṇakutūhala* [KaKu1991, p. 38] *tripraśna* chapters.

Rājamṛgāṅka 3.12–14 :	*Karaṇakutūhala* 3.7 :
दिनार्धं दिनपातोनं प्राक्कपाले नतं हि तत् । दिनार्धोने दिनगते नतं प्रत्यक्कपालजम् ॥१२॥ Half the day decreased by the elapsed [part] of the day is the *nata* [from the meridian along the equator] in the eastern hemisphere; when the past [part] of the day is decreased by half the day, [that] is the *nata* produced in the western hemisphere. रात्रिशेषघटीयुक्ते दिनार्धे प्राक् नतं मतम् । प्रत्यक् नतं दिनार्धेऽथ रात्रियातघटीयुते ॥१३॥ When the half-day is added to the *ghaṭikās* of the remainder of the night, the *nata* is considered eastern; when the half-day is added to the *ghaṭikās* of the past [part] of the night, then the *nata* is western. नतोनितं दिनदलम् उन्नतं तत्प्रकीर्तितम् । नतात् तथोन्नतात्[1]षड्भिः गुणिताज्ज्यादि साधयेत् ॥१४॥ The half-day decreased by the *nata*: that is said to be the *unnata*. From the *nata* and likewise from the *unnata*, multiplied by six [to turn *ghaṭikās* into degrees], one should determine the R sin etc.	चरपलयुतहीना नाडिकाः पञ्चचन्द्राः द्युदलमथ निशार्धं याम्यगोले विलोमम् । द्युदलगतघटीनामन्तरं तन्नतं स्यात् नतरहितदिनार्धञ्चोन्नतं जायतेऽत्र ॥७॥ Fifteen *ghaṭikās*, increased or decreased by the *vighaṭikās* of the half-equation of daylight, are half the day or half the night [respectively]; when [the sun is] in the southern [hemi]sphere, vice versa. The difference of half the day and the elapsed *ghaṭikās* [of the day] should be the *nata*; and [when] half the day [is] decreased by the *nata* the *unnata* is produced here.

$$h_n \approx \frac{h_0 \cdot \left(10 + \frac{5}{6}\left(\frac{1}{2} \cdot \frac{\omega^{(\mathrm{vigh})}}{10\, s_0}\right)^2\right)}{10 + \omega^{(\mathrm{gh})}}, \qquad h \approx \frac{h_0 \cdot \left(10 + \frac{5}{6}\left(\frac{1}{2} \cdot \frac{\omega^{(\mathrm{vigh})}}{10\, s_0}\right)^2\right)}{10 + \omega^{(\mathrm{gh})} - \frac{50n^2}{n^2 + 900}} \tag{7}$$

As discussed in [Plo2016], "half the day-length minus five" is equivalent to $10 + \omega$ *ghaṭikās*, and "the half-equation of daylight divided by the first [ascensional] difference" is $\omega^{(\mathrm{vigh})}/(10\, s_0)$ (see also [MP, section 2.1.5]). Bhāskara

Table 3: Verses on algebraic approximations to the gnomon shadow and hypotenuse in the *Rājamṛgāṅka* [RaBh1987, p. 13] and *Karaṇakutūhala* [KaKu1991, p. 39] *tripraśna* chapters.

Rājamṛgāṅka 3.19–22 :	*Karaṇakutūhala* 3.8–10 :
दिनमानं विदिक्कृत्वा ततः कुर्यान्नवाहतम् । तत्कालनतवर्गाप्तं विदिग्दिनशतांशयुक् ॥१९॥ Having made the day-length minus ten, then one should make [it] multiplied by nine [and] divided by the square of the *nata* at that time, plus a hundredth part of the day-length minus ten.	दिनदलं विशरं खहरो भवेत् नतकृतिः पृथगभ्रशराहता । खखनवाढ्यपृथक्स्थितया हृता खहरतः पतिताभिमतो हरः ॥८॥ Half the day-length minus five [*ghaṭikās*] should be the noon “divisor”. This square of the *nata* is separately multiplied by fifty and divided by [the same square] increased by 900 [in] a separate place; [the quotient is] subtracted from the noon divisor; [the remainder] is the divisor desired [for the intended time].
विज्ञेयो गुणकः सोऽथ व्येको हर इति स्मृतः । शङ्कुवर्गगुणं कुर्याद् ऐक्यं गुणकहारयोः ॥२०॥ मध्यभागुणकाघात- वर्गयुक्तं तु तद्भवेत् । साधयित्वाथ तन्मूलं हरेण विभजेत् ततः ॥२१॥ That is to be known [as] the “multiplier”; now, [that] minus one is considered the “divisor”. One should make the sum of the multiplier and divisor and it should be multiplied by the square of the gnomon, added to the square of the product of the noon shadow and the multiplier.Having now determined the square root of that, one should divide by the divisor.	अथ नतं यदि पञ्चदशाधिकं दिनदलात्पतितं स हरस्तदा । प्रथमखण्डहृतं दलितं चरं स्वगुणितं स्वषडंशविवर्जितम् ॥९॥ Now if the *nata* is greater than fifteen [*ghaṭikās*], when subtracted from half the day, it is the divisor at that [desired] time. The half-equation of daylight divided by the first [ascensional] difference, halved, multiplied by itself, is decreased by its own sixth part. Added to ten [and] multiplied by the equinoctial hypotenuse, [it is] the “multiplier”, [which] divided by the divisor [is] the hypotenuse beginning with digits.
अङ्गुलाद्यत्र यल्लब्धं सेष्टच्छाया भवेत् स्फुटा । विषुवत्कर्णवद् भात इष्टकर्णं प्रसाधयेत् ॥२२॥ The quotient here in digits etc., should be the accurate desired shadow. From [this] shadow one should calculate the desired hypotenuse, as [for] the equinoctial hypotenuse.	दशयुतं पलकर्णहतं हृतिः हरहता श्रवणोऽङ्गुलपूर्वकः । रवियुतोनितकर्णहतेः पदं द्युतिरिनद्युतिवर्गयुतेः श्रुतिः ॥१०॥ The square root of the product of the hypotenuse [with itself, separately] increased and decreased by twelve, is the shadow; [the square root] of the sum of the squares of twelve and the shadow is the hypotenuse.

also mentions that when the *nata* is greater than 15 *ghaṭikās*, the *unnata* itself can be used as the divisor instead of the more complicated expression involving the square of the *nata*; but we do not investigate this variation in this paper.

The rather cumbersome complexity of these "easy" methods for avoiding the explicit computation of R sines justifies itself when we consider their impressive accuracy, as revealed in the sample graphs of Figure 4. In these figures the shadow-length s in digits resulting from each formula, as well as from a trigonometrically exact version, is graphed against degrees of declination δ for specified values of ϕ and n. For purposes of comparison, the hypotenuse-length produced by Bhāskara's rule has been converted to shadow-length by applying the right-triangle relation $s = \sqrt{h^2 - 12^2}$. (Although it might seem more intuitive to graph s against n for given values of ϕ and δ, as the examples in Figure 5 illustrate, the large variation in shadow-length during the course of a day makes it almost impossible to distinguish among the different function plots for any range of n-values greater than a few degrees.)

The two authors continue their topic with the immediately following *anuṣṭubh* and *drutavilambita* verses, respectively, quoted in Table 4. Here they describe rules for computing the *nata* n if the corresponding shadow-hypotenuse h and the half-equation of daylight ω are known.

These rules too are algebraic approximations that avoid "the procedure of finding R sines". The *Rājamṛgāṅka* rule is equivalent to the expression

$$n^{(\mathrm{gh})} \approx \sqrt{\frac{9(20 + 2\omega^{(\mathrm{gh})})}{\dfrac{h}{h - h_n} - \dfrac{20 + 2\omega^{(\mathrm{gh})}}{100}}}, \tag{8}$$

while the *Karaṇakutūhala* version can be expressed symbolically as follows:

$$n^{(\mathrm{gh})} \approx \sqrt{\frac{900\left(10 + \omega^{(\mathrm{gh})} - \dfrac{h_0}{h}\left(10 + \dfrac{5}{6}\left(\dfrac{1}{2}\cdot\dfrac{\omega^{(\mathrm{vigh})}}{10\, s_0}\right)^2\right)\right)}{50 - \left(10 + \omega^{(\mathrm{gh})} - \dfrac{h_0}{h}\left(10 + \dfrac{5}{6}\left(\dfrac{1}{2}\cdot\dfrac{\omega^{(\mathrm{vigh})}}{10\, s_0}\right)^2\right)\right)}}. \tag{9}$$

Both of these formulations appear to be the result of some sort of inversion of the rules represented by equations (6) and (7). Their even greater effectiveness can be estimated from the sample graphs in Figure 6, plotting n in *ghaṭikās* against δ in degrees for given values of ϕ and h. As the graphs illustrate, the algebraic approximations agree extremely closely with each other and with the trigonometrically exact expression.

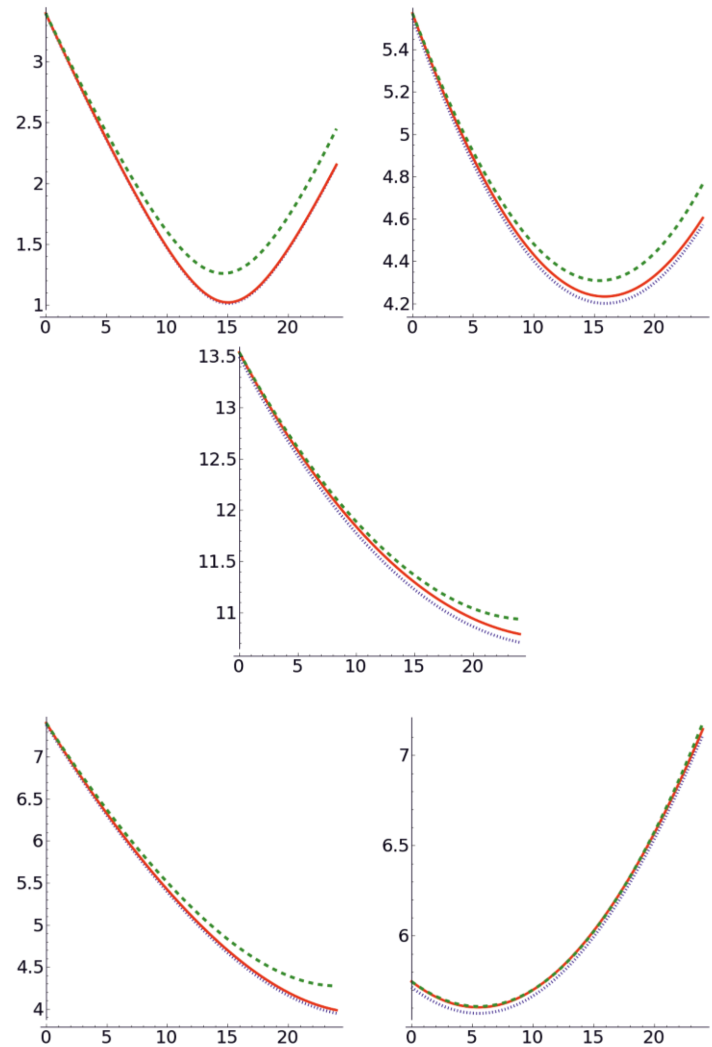

Figure 4: Comparisons of the shadow-approximations of Bhojarāja (solid red) and Bhāskara (dashed green) with a trigonometrically exact version (dotted blue). The graphs plot shadow-length s in digits against declination δ from 0 to 24 degrees, for various values of latitude ϕ and *nata* n, also in degrees. Left to right, top to bottom: $\phi = 15$, $n = 5$; $\phi = 15$, $n = 20$; $\phi = 20$, $n = 45$; $\phi = 25$, $n = 20$; $\phi = 5$, $n = 25$.

Table 4: Verses on algebraic approximations for determining the *nata* from the shadow-hypotenuse in the *Rājamṛgāṅka* [RaBh1987, p. 13] and *Karaṇakutūhala* [KaKu1991, p. 39] *tripraśna* chapters.

Rājamṛgāṅka 3.23–25 :	*Karaṇakutūhala* 3.11–12 :
इष्टकर्णोऽन्तरहृत इष्टमध्याह्नकर्णयोः । दिगूनद्युशतांशेन रहितो भाजको भवेत् ॥२३॥	श्रुतिविभक्तहृतिस्तु हरो भवेत् स पतितः खहरादवशेषकम् । पृथगिदं खखनन्दहतं हरात् खविषयैरवशेषविवर्जितैः ॥११॥
The desired hypotenuse divided by the difference of the desired and midday hypotenuses, decreased by a hundredth part of the day minus ten, should be the divisor.	The dividend divided by the hypotenuse should be the divisor; it is subtracted from the noon divisor. One should divide this remainder, separately multiplied by 900, by 50 decreased by [the same] remainder.
दिनप्रमाणाद् रहितात् दिग्भिर्नवभिराहतात् । भाजकेनाथ यल्लब्धं तत्पदं नतनाडिकाः ॥२४॥	फलपदं हि नतं यदि शेषकं दिगधिकं हर एव तदोन्नतम् । इति कृतं लघु कार्मुकशिञ्जिनी ग्रहणकर्म विना द्युतिसाधनम् ॥१२॥
From the amount of the day decreased by ten [and] multiplied by nine, whatever is now the quotient with the divisor, the square-root of that should be the *ghaṭikās* of the *nata*.	The square-root of the result is the *nata*. When the remainder is greater than ten, the divisor itself is that *unnata*. Thus the easy determination of the shadow is made without the procedure of finding *R* sines.
ताश्च मध्याह्नतः शोध्या दिनार्धघट्यो गतागताः । इष्टच्छायेष्टकर्णाभ्याम् एवं समयसाधनम् ॥२५॥	
And those are subtracted from the midday *ghaṭikās* of half the day past/future. Thus from the desired shadow and desired hypotenuse is the determination of time.	

3 Issues for future analysis

Of all the many questions we could ask about these ingeniously derived approximation formulas, the most interesting are probably the following:

- Exactly how did Bhojarāja and Bhāskara each invent, test and refine their formulas?
- Exactly what influence, if any, did the *Rājamṛgāṅka*'s *tripraśna* chapter have on the development of the *Karaṇakutūhala*'s?

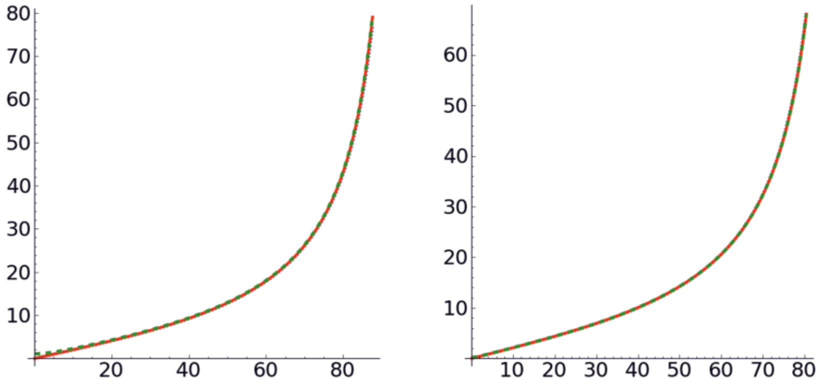

Figure 5: Unsuccessful comparisons of the shadow-approximations of Bhojarāja (solid red) and Bhāskara (dashed green) with a trigonometrically exact version (dotted blue). The graphs plot shadow-length s in digits against *nata* n in degrees, for various values of latitude ϕ and declination δ, also in degrees. The small differences between the plotted lines have become invisible due to the large variation in s as n changes. Left to right: $\phi = 20$, $\delta = 20$; $\phi = 5$, $\delta = 5$.

Unfortunately, neither of these questions is likely to be answerable in any conclusive way. We do know [CESS, A4.299] that Bhāskara's great-great-great-grandfather Bhāskara Bhaṭṭa received the title *vidyāpati*, presumably as a court astronomer, from Bhojarāja himself. So it would be only natural for the highly learned Bhāskara to be acquainted with Bhojarāja's *karaṇa* via a family library. But a better answer will have to await a thorough comparison of the two handbooks and the earlier texts that might have influenced them both.

Similarly, we can make some general remarks about reconstructing the derivation of these approximations, building on the patterns tentatively sketched in [Plo2016]. For one thing, the designation of "multipliers" and "divisors" to facilitate the statement of complicated trigonometric rules for relating time to gnomon shadows is quite common in the *tripraśna* chapters of astronomical treatises, and may have inspired their use in these innovative rules. The study and reconstruction of other similar approximation formulas in Sanskrit astronomical works may help us identify more clearly the nature and content of the medieval numerical-analysis "toolkit" that mathematicians such as Bhāskara relied on to create them.

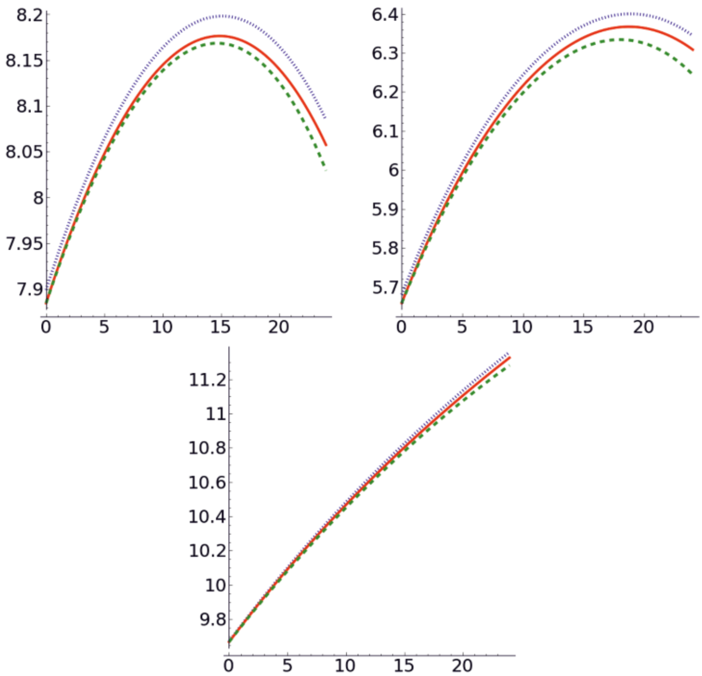

Figure 6: Comparisons of the *nata*-approximations of Bhojarāja (solid red) and Bhāskara (dashed green) with a trigonometrically exact version (dotted blue). The graphs plot *nata* n in *ghaṭikās* against declination δ from 0 to 24 degrees, for various values of latitude ϕ in degrees and hypotenuse h in digits. Left to right, top to bottom: $\phi = 10$, $h = 18$; $\phi = 15$, $h = 15$; $\phi = 25$, $h = 25$.

Part VII

Reach of Bhāskarācārya's works and its pedagogical importance

जयति जगति गूढानन्धकारे पदार्थान्
जनघनघृणयायं व्यञ्जयन्नात्मभाभिः ।
विमलितमनसां सद्वासनाभ्यासयोगैः
अपि च परमतत्त्वं योगिनां भानुरेकः ॥

jayati jagati gūḍhān andhakāre padārthān
janaghanaghṛṇayāyaṃ vyañjayann ātmabhābhiḥ |
vimalitamanasāṃ sadvāsanābhyāsayogair
api ca paramatattvaṃ yogināṃ bhānur ekaḥ ||

Triumphant is the Sun, who, out of great compassion for the people, all by himself with his brilliant rays making manifest all the objects engulfed by darkness, reveals the supreme truth to the *yogis*, who have purified their minds by the practice of austerities and *yoga*.

Persian translations of Bhāskara's Sanskrit texts and their impact in the following centuries

S. M. Razaullah Ansari*

1 Introduction

The translation movement of scientific texts initiated during the early Abbasid period (750-950) in West and Central Asian countries during the Islamic Middle Age was naturally transmitted to medieval India during the Sultanate and Mughal periods, when learned scholars seeking patronization thronged to the courts of Indian Sultans, Mughal emperors, local rulers and nobles (*Umarā*'). Some of those rulers and their courtiers were themselves scholars, who had keen interest to promote rational sciences, mathematics and astronomy in particular, and to acquire the medieval sciences in general by supporting the translations from Sanskrit sources, as their forefathers practised in Islamic countries earlier. I may mention the example of Sultan Fīrūz Shāh Tughlaq (r. 1351-1388), who was keenly interested in astrolabes, [Sar2000, pp. 129–147] and ordered the translation of Sanskrit astrological material, of which two texts are known.

1. *Dala'il-i Fīrūzshāhī* (in verse), translated by the poet 'Izzuddīn Khālid Khānī (or Khafī). It dealt with the rising and setting of the seven planets and their good and evil import. The tract has been reported by 'Abdul Qādir Badāyūnī who saw it in Lahore in 1591 [Bad1973, p. 332].[1]

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[1] 'Abdul Qādir Badayūnī (1540-1615) was a famous translator of Sanskrit classics, for instance *Mahābhārata, Rāmāyaṇa* and a History of Kashmir into Persian at the insistence of Emperor Akbar.

K. Ramasubramanian et al. (eds.), *Bhāskara-prabhā*, Sources and Studies in the History of Mathematics and Physical Sciences, https://doi.org/10.1007/978-981-13-6034-3_18

2. *Tarjumah-i Bārāhī* or *Kitāb Bārāhī Sanghtā,* a translation of Varāhamihira's *Bṛhatsaṃhitā* by 'Abdul 'Azīz Shams Thānesarī, about half a dozen manuscripts of which are known in India.[2]

However, since in this paper, I wish to confine myself with the Mughal period only, a few words about the *Translation Bureau* founded by Emperor Akbar (1556-1605) will be appropriate.

2 Akbar's Translation Bureau

The Bureau was founded in Fatḥpur Sīkrī (Agra). The main objective of Akbar was to promote exchanges between the Persian-knowing Muslims and Hindu intellectuals and simultaneously to satisfy Akbar's own spirit of acquisition of learning. Already in 1574, Akbar commissioned the writer and historian 'Abdul Qādir Badāyūnī (1540-1615) to translate into Persian from Sanskrit (SK) the "Tales of Thirty-two Statues of the Throne of King Vikramaditya" (eleventh century). The actual title in SK is *Singhasan Battisī*, and in Persian *Nāma'-i Khirad Afzā* (Book on Enhancement of Wisdom) [Abb1992, p. 229] and [Riz1975].

During the subsequent years, however, the most popular epics of Sanskrit literature the *Mahābhārata* (MB) and the *Rāmāyaṇa* were also commissioned for translation into Persian. The former was commenced already in 1583 by Naqīb Khān with the co-operation of 'Abdu'l Qādir Badāyūnī, and assisted by Hājī Sultān Thānesarī and Mullā Shīr. Akbar ordered also Abu'l Fayḍ Fayḍī to versify the Persian text. In short that joint effort resulted into the complete translation of MB into Persian in 1587 with a detailed *Muqaddama* (Foreword) by Abu'l Faḍl. It was entitled as *Razm Nāma.*[3] The second epic, *Rāmāyaṇa,* was translated in 1584 by Badāyūnī himself. It suffices to list below only translations of other Sanskrit texts with some important information, if any.

Rājataraṅgiṇī (*The History of Kashmir*) by Kalhaṇa Paṇḍit was translated by Shāh Muḥammad Shāhābādī directly from Sanskrit into Persian and who presented it to Emperor Akbar on his first visit to Kashmir in July 1589. The Emperor ordered Badāyūnī to polish the language, which the latter carried

[2] Cf. [JS1985, pp. 161–169] and the recent essay by Eva Orthmann [Ort2017].

[3] For a detailed account of the team work see [Abb1992, pp. 232–233] or the original article of Rizvi [Riz1975].

out and completed the revised Persian text in 1591.[4] The *Kathāsaritsāgara* ('Ocean of the Streams of Stories') is a famous eleventh-century collection of Indian legends, fairy tales and folk tales as retold in Sanskrit by a Shaiva named Somadeva and contains 22,000 *ślokas.* The Persian translation of some parts of this Collection had been done during Sultan Zainul 'Ābidīn in Kashmir, entitled *Baḥrul Asmā'.* Akbar ordered Badāyūnī in 1594 to revise that archaic text into simple Persian for the sake of understanding and to complete the remaining part also.[5]

Kalila wa Dimna is another world-wide known collection of fables in the form of animal stories, originally written in Sanskrit as *Pañcatantra.* The fables were meant for princes to conduct their life wisely. They were translated into Arabic in 750 CE by Ibn al-Muqaffa (d. 756/759), into Middle Persian in 570 CE and in classical Persian in the fifteenth century by Ḥusain Wā'iẓ Kāshifī. Akbar ordered Abu'l Faḍl to write an easily understandable Persian language version, which was completed by Abu'l Faḍl in 1588, with the title *'Ayār-i Dānish* (Fineness of Wisdom).

It may also be noted that Akbar was also keenly interested in world religions, philosophy and history. He requested in his letter to the Portuguese king to send to India an embassy of philosophers and historians particularly. 'Abdus Sattār bin Muḥammad Qāsim Farishtah (the historian) was ordered by Akbar to learn Portuguese and Latin, in order to translate into Persian European philosophy and religious writings [Sal1993, pp. 31–32]. The resulting codex is entitled *Thamra al-Filāsafa* (Fruits of Philosophers). It contains also a biography of Jesus.[6] Finally to add is that the Translation Bureau did not neglect translation of scientific texts. For instance, a team of both Muslim and Hindu scholars headed by Fatḥullāh Shīrāzī began translating *Zīj-i Ulugh Beg* into Sanskrit. In fact the reception of sciences also continued even up to the end of the Mughal period in the nineteenth century. For scientific works in Sanskrit and their translation or adaptation, see [Ans2009, ch. 8, pp. 251–274]. However, in the following, I am concerned only with the translation of Bhāskara's works.

[4] See [Ogu2010, pp. 33–37]. The paper is an excellent critical study of Persian translations of the *Rājataraṅgiṇī*: one during Akbar's time, another by Ḥaidar Malik in 1618, and an anonymous abridged text during Jahangir's reign.

[5] [Abb1992, p. 240]. See also [Bad1973, pp. 415–416].

[6] Sattār was actually a pupil of Fa. Jerome Xavier (d.1716), who spent twenty years in the service of Emperor Akbar and later was patronised by Emperor Jahāngīr. Sattār and Xavier translated mostly books of Christianity, see Gulfishan Khan [Kha1998], chapter 4 on Religion.

3 Fayḍī, the translator of *Līlāvatī*

The *Līlāvatī* was translated by the poet and scholar Abu'l Fayḍ Fayḍī (b.1547–d.1596 in Lahore) by the order of Emperor Akbar. Abu'l Fayḍ Fayḍī joined the court of Emperor Akbar in 1568. His father Shaikh Mubārak (d. 1593)[7] instructed Fayḍī in traditional subjects such as religion, philosophy, grammar, science, and medicine. Faiḍī served as tutor to Akbar's three sons, Salīm (later Emperor Jahāngīr, r.1605–27), Murād, and Dānīāl. Fayḍī was the elder bother of Abu'l Faḍl, the chronicler of the reign of Emperor Akbar. He achieved highest recognition in both rational sciences (*'Ulūm-i 'Aqliya*) and traditional sciences (*'Ulūm-i Naqliya*). It has been reported that he had thousands of books in his library. He was bestowed by Akbar the title of Poet Laureate (*Malik al-Shurā'*) in 1588, with *nom de plume Fayyāḍī.* Although he compiled about 100 poetic works, unfortunately only a few survived.

In 1583 Fayḍī was ordered by the Emperor Akbar to come to Lahore to participate in his Bureau of Translation. He was also commissioned to translate the Sanskrit text of *Līlāvatī* by Bhāskarācārya which he completed in 1587. The following anecdote about his expertise of Sanskrit language is noteworthy [Āzā1922, pp. 100].

> For several years he acquired knowledge of *Shastras* [Sanskrit books] as a disguised Brahman in Banaras (now Varanasi). When his teacher came to know about it, he advised Fayḍī not to translate into Persian the *Gāyatrī Mantar*[8] and the four *Vedās.* Fayḍī agreed and kept his promise.

This legendary expertise is not to be taken literally. It might have been motivated by Fayḍī's competition with other translators of Sanskrit works, for instance 'Abdul Qādir Badāyūnī, the translator of the *Rāmāyaṇa* into Persian, and a member of the team of scholars commissioned to translate the *Mahābhārata* (see 2 above).

Before I discuss his translation of the mathematical book *Līlāvatī,* it will be appropriate to list some of his other writings as well. Fayḍī was also commissioned to translate into Persian the romantic story of emperor Nala and princess Damayantī as related in the *Mahābhārata*, with the title, *Nal wa Daman.* It had been a very popular story containing themes such as love, de-

[7] Shaikh Mubārak (1505-1593) had been an expert on classics of rational science, e.g. Ibn Sīnā's works on natural philosophy (*Shifā'* and *Ishārāt*), *Almagest* by Ptolemy and Naṣīruddīn al-Ṭūsī's 'Non-Ptolemaic model of planetary motion', as presented in his *Memoir on Astronomy* (Arabic Title: *Al Tadhkira fī 'ilm al-Hay'a,* three Mss of which are extant in India to-date: one in Hyderabad (SL) and two in AMU (MA Lib.)).

[8] The *Gāyatrī Mantra* is a highly revered mantra from the *Ṛgveda,* dedicated to the Sun deity. *Gāyatrī* is the name of the *Vedic* meter in which the verse is composed.

ceit and war between Hindu deities and Nal. Fayḍī translated it from Sanskrit into Persian in the style of a *Mathnavī.* It contains about 4000 verses.[9]

Besides Persian and Sanskrit, Fayḍī had actually a great command on the Arabic language. He wrote an exegesis of the Qur'ān in Persian, completed in 1594, with the title *Sawāṭi al–Ilhām* (Illumination of Inspiration).[10]

The remarkable feature of this text is that he used through out only 13 *undotted* letters out of the 28 total letters of the Arabic alphabet. His book on ethics with the title *Mawārid al-Kalām* (Stages of Words) contained also only undotted Arabic letters. In passing I may mention that Fayḍī compiled in about 1585 his *Dīwān* (Collections of Poems), which contained about 6000 verses. Its title is *Ṭabāshīr al-Ṣubḥ* (Prelude to Dawn). It comprises all genres of Persian poetry, viz., panegyrics (*qaṣā'id*), *ghazals* (poems on love), *rubā'īs* (two couplets), *mathnavīs* and elegies.[11]

3.1 Survey of manuscripts of the Līlāvatī

The Sanskrit original text was printed in Calcutta in 1932. It was translated into English by J. Taylor and published from Bombay in 1816 [Tay1816], and by H. T. Colebrook [Col1817]. The Persian text with the title *Nuskha'-i Līlāvatī* (Manuscript of *Līlāvatī*) was published lithographically from Calcutta in 1827 and 1854. I may mention that Taylor secured from the well known Parsi scholar Mullā Fīroz of Bombay a copy of Fayḍī's translation and tried to assess the Persian text. His comments are as follows [Tay1816, Intro. p. 2]:

> His translation possesses that general accuracy which might be from a person of Fayzi's talents and knowledge, aided by such eminent mathematicians as his own situation, or the influence of his Royal Patron could obtain. It is however often very obscure, and in several places there are considerable omissions…. The chapter on indeterminate problems and on transposition are altogether omitted.

[9] Interestingly he talks about three thematic oppositions in the text. The first opposition is love and intellect (*'Aql*), the second opposition is between love and beauty (*Ḥusn*) and the third opposition is between *'Ishq and Junūn (frenzy);* see for details the article in *Wikipedia,* accessed on 15.7.2017. See also [AS2006, pp. 109–141], not available to me presently.

[10] For a short write up and commentaries on this exegesis, see [Iṣl2002, pp. 114–118]. The author deals with works of Sheikh Mubārak and his two sons Fayḍī and Abu'l Faḍl. For Fayḍī, see [Iṣl2002, pp. 112–121].

[11] See details in the article on Fayḍī <www.iranicaonline.org/article/fayzi-abul.fayz>, author, Munibur Rahman, accessed on 15.7.17.

Note that the second sentence on obscurity is contradictory to the first sentence above. In fact in no other manuscript of the *Līlāvatī* have I found any marginal comment on the alleged obscurity of the Persian text. Moreover, Fayḍī and other successors were interested only in the commercial arithmetic of the *Līlāvatī*, which explains the omissions on such theoretical problems as indeterminate analysis of pure mathematics. In any case only a verbatim comparative study of Fayḍī's text with that of the original Sanskrit text can decide whether Taylor understood the Persian text correctly or not.

Here I do not go into the mathematical details of the text. Rather I survey the available manuscripts of Fayḍī's translation, in order to stress its impact on the scholars of the Indian subcontinent. I shall confine myself to the extant manuscripts in Indian and Pakistani libraries. For foreign collections the reader may consult the *Catalogue* by Rosenfeld and İhsanoğlu [RE2003].[12] The various manuscripts are listed by city (library symbol); full form of symbols for the libraries/Mss collections are listed in the Appendix.

One each in Calcutta (ASL), Deoband (MDU), Hyderabad (IAU, SCL), London (BM), Lucknow (MNU), Patna (KhB), Srinagar (IL) and Tonk (TL), while 2 Mss each in Lucknow (RML) and Patiala (PL), 4 in Aligarh (HG, Sub, Sul, QD) and 3 in Rampur (RL), and 5 in New Delhi (JHL), the total = 25 Mss.[13] Besides these, 12 manuscript copies are extant in Pakistani libraries, out of which 2 Mss are in Lahore (PUL) and 1 in Karachi (NM) [Mon1983, pp. 214-215]. That is, presently 37 manuscripts are extant on the subcontinent. This number is not very large, but the book was available in print in the nineteenth century and therefore hand-written copies fell out of popular use.

3.2 Other translations

1. A Persian translation of the *Līlāvatī* which has been overlooked by historians of mathematics generally is presumably by an Uzbek scholar, Ibn Yalb, entitled *Mirāt al–Ḥisāb* (Mirror of Arithmetic). The unique Persian manuscript is in a codex No. 6230/1, with 43 ff in the manuscript collection of the Institute of Oriental Studies, Tashkent. This manuscript copy was scribed in AH 1231/1816 CE according to internal evidence. Noth-

[12] This catalogue is actually a revised and updated edition of Vol. 2 of the Catalogue by Matvievskaya and Rosenfeld [MR1983]: *Mathematiki Astronomy Musul'manskovo i Crednevskov'ya i ix Trudy* (VIII-XVII Centuries), 3 Volumes in Russian, Izdatel'stvo Nauka, Moscow, 1983. It is an excellent source of information on Islamic exact science. Thanks to the late Prof. Boris Rosenfeld (Moscow), who presented this set to me.

[13] Cf. [Qas2014, p. 234]. Qasmi missed 3 manuscripts: one each in Aligarh and Patiala and one in Tonk.

ing is known about the author. However according to the date given at the beginning of the manuscript, AH1098/1687-88 CE, the author lived presumably in the seventeenth century.

I am happy that I could secure the paper by Hanifa R. Muzafarova (in Russian) on this Persian translation, the English translation of which by Shamim Bano (Delhi) was edited with annotations by me.[14] Ibn Yalb's translation comprises two parts. Part I (pp. 138–151) deals with the *Līlāvatī* and is very detailed with mathematical contents. Part II is the translation of Bhāskara's *Bījagaṇita,* of which Muzafarova mentions only the topics, viz., the system of linear equations and their solutions.

2. H. J. J. Winter (London) and Arshad Mirza (Delhi) translated into English the incomplete Fayḍī's Persian version of the *Līlāvatī* extant in the John Rylands Library in Manchester, dated 1720 CE. The partial text concerns commercial arithmetic [WM1952, pp. 1–10].[15]

3. The translation by Qaḍī Ḥasan b. Khwājah Ṭāhir bin Khwājah Muḥammad is entitled *Tarjumah Līlāvatī.* The undated manuscript is in Hamidia Library, Bhopal.[16]

4. Mednīmal bin Dharam Narāyan bin Kalyānmal (of Itāwah and of Hindu caste of Kāyasth) compiled in Persian a treatise on arithmetic based on the *Līlāvatī.* The title is *Badā'i al-Funūn* (Novel Techniques). The year of writing is 1664—the sixth regnal year of Emperor Aurangzeb. Its ten manuscripts copies are extant in Calcutta (AS) and in the libraries of Hyderabad in India.[17] Four manuscript copies are in Pakistan.[18]

5. Ānand Kāhin bin Hemrāj Kāyesth from Guwaliar wrote a tract *Ḥisāb Nāmah.* Its two manuscripts are extant in India, one in Suleman collection, M.A. Library (AMU, Aligarh) and another in Cultural Academy, Srinagar (Kashmir), in which the title of the tract is *Mukhtaṣirī dar Ḥisāb* (Summary of Arithmetic). According to the cataloguer Muḥammad Ibrāhim, 'this tract is actually a selection from *Līlavatī,* written in

[14] [Muz1985, pp. 135–152] is listed by Rosenfeld and İhsanoğlu (2003) in their *Catalogue* at [RE2003, No. 1136, p. 365]. But they assumed wrongly that he was an Indian mathematician.

[15] The title of their work is '*Concerning the Persian Version of Lilavati*'. The title is misleading, as if they commented or evaluated Faizi's text. But this is not true. The authors did not evaluate or comment, rather they translated the incomplete text just literally.

[16] See [Ham1986, p. 72, No. 4/872].

[17] [Qas2014, p. 235] lists the libraries and details of manuscripts, see also [Rah1982, p. 410]. This catalogue comprises separate sections on Mathematics, Astronomy Medicine, Physics, etc.

[18] For details, see [Mon1983, pp. 179–180].

1820-21' [Ibr1986, pp. 357–358]. There are extant three manuscripts in Pakistan: One in Karachi (NM) scribed in 1867 and two in Lahore (PUL) [Mon1983, p. 192]. The scribe states in the colophon of one Ms in Lahore, that it is '*Līlāvatī* in Persian prose, ...on arithmetic and mensuration and that this translation was printed in Fakhrul Muṭābi Press in Delhi'.[19]

6. A late Persian translation of the *Līlāvatī* by Ḥasan 'Alī Hātiqī was printed in 1862 at Dilkushā Press (probably at Delhi), a copy of which is extant in the University Persian Collection, Aligarh Muslim University, Aligarh.
7. Fayḍī's Persian text was also translated into Urdu in 1844 by Sayyid Muḥammad Ghāfil ibn S. Mahdī 'Alī during Emperor Bahādur Shāh's reign (1837-1857). Its Ms is in Aligarh (Sul. 46/2). The title is *Sharḥ Līlāvatī* (Commentary on *Līlavatī*), scribed in 1844 with 89 ff [Anṣ1995, p. 28] and [Kho1995, p. 99].
8. Another Urdu translation was carried out by a Hindu scholar Debi Chand in 1855 [Anṣ1995, p. 28].

These fresh Persian translations and also Urdu versions of the Persian text of Līlāvatī, listed above, indicate that just two years prior to 1857—when the first War of Independence broke out in the Subcontinent, in short then a very difficult time—the patronization of scholars continued and the work on a fresh translation did make sense. The same was the case also with Urdu translations of astronomical-astrological works, which are also listed by Anṣārullāh in his paper.

4 *Bījagaṇita*

The *Bījagaṇita* was translated into Persian in ca. 1634-35 by 'Aṭāullāh Rushdī bin Aḥmad Ma'mār. His father was the architect of the Taj Mahal in Agra. 'Aṭāullāh was also an architect (*Muhandis*). He designed in Aurangābād the mausoleum of Dilras Bāno Begum Rābi'a Daurānī (d. 1659), wife of Emperor Aurangzeb b. Shāhjahān (r. 1658-1707).[20] He was the elder brother of Luṭfullāh Muhandis, who became a famous mathematician of medieval India. 'Aṭāullāh is known to have written two mathematical works.

[19] This manuscript is in the Shīrānī Collection of Lahore (PUL), quoted by [Anṣ1995, p. 27].

[20] This information is given in the inscription on a bronze plate, installed on the gate of the mausoleum. The name of its maker is given as Hībat Rā'i. Chughta'i (p. 31) cites *Gazetteer of Aurangabad* (1883) as his reference. The plate is dated AH 1072/1661-62 CE [Chu1957].

1. *Khulāṣatul Rāz* (Essence of Mystery [of Arithmetic]) is a book in Persian on arithmetic, algebra and mensuration. It is in verse form and is dedicated to Dārāshikoh (d. 1657), eldest son of Emperor Shāhjahān. Actually, this tract is an abridged translation of the popular and famous *Khulaṣat al-Ḥisāb* (The Essence of Arithmetic) in Arabic by Bahā'uddīn al-'Āmilī (1547-1622), which was used as a textbook in madrasas of medieval India.

2. *Tarjuma'-i Bījganit* is the Persian translation of Bhāskara's original text in Sanskrit, entitled *Vījagaṇita* or *Bījagaṇita.*[21] It was dedicated by the author 'Aṭāullāh Rushdī to Emperor Shāhjahān. E. Strachey translated into English this Persian version [Str1813].[22] Noteworthy is that 'Aṭāullāh praises Bhāskarācārya in the introduction of his text as follows:[23]

   > In the science of calculations it [Bhāskara] is a discoverer of wonderful truth and nice subtleties and it contains useful and important problems which are not mentioned in the *Līlāvātī* nor in any Arabic or Persian book. I have dedicated the work to emperor Shahjahan and I have arranged it according to the original in an introduction and five books.

   In the following, I confine myself with enumerating only the extant manuscripts of *Bījganit* on the subcontinent. The following 11 manuscripts of Persian texts are extant in Indian libraries to-date according to my survey: One manuscript each is extant in Banaras (BHU), Deoband (MDU) and Rampur (RL), two each in Hyderabad (SL, OUL) and Lucknow (MNU, MWL), and 4 Mss at London (BM, IO).[24] I am including the Mss in the libraries of the British Museum and the India Office (London) here, since they were extant actually in India before their transfer to London.

Besides the above mentioned translation, the following two translations may be noted also:

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[21] See for Bhāskara and his writings [SS1966, p. 20], [Sto1972, pp. 5 and 15] and [BīGa2009, pp. 3–301]. According to Michio Yano (Ed.), '…it can be called the most reliable and most useful edition hitherto published, since Hayashi has meticulously collated the seven most recently published editions and added seven helpful appendices, …'

[22] The English translation of the original Sanskrit text was carried out by Colebrooke [Col1817].

[23] Cited by Strachey (*loc. cit.*) in the beginning of his translation of the actual text of *Bījganit,* [Str1813, p. 28]. I have compared this translation with the Persian original and found it quite correct.

[24] [Str1813, p. 5] did not cite any manuscript from Indian libraries. [Qas2014, p. 235] and [Rah1982, p. 391] list six and five Mss in India respectively. For the four manuscripts abroad at Cambridge, London (Royal Asiatic Society), Munich and Paris, see [RE2003, pp. 372-373].

1. The Indian scholar Muḥammad Amīn bin M. Sa'īd al-'Alawī translated into Persian the Sanskrit text of *Bījagaṇita* with the title, *A'jāz al–Ḥisāb* (written in 1661) during the reign of Emperor Aurangzeb. It is extant in the Raza Library (Rampur).[25]
2. An anonymous translation with the Persian title is *Badr al–Ḥisāb* (written in 1688–89) and is extant in the State Central Library (Hyderabad). It is scribed by Raghūnāth in Dhaka (now in Bangladesh).[26]

5 *Siddhāntaśiromaṇī*

The *Siddhāntaśiromaṇī* was composed by Bhāskara in 1150. This treatise on theoretical astronomy was translated into Persian in 1797 by Ṣafdar 'Alī Khān bin Muḥhammad Ḥasan Khān, who dedicated it to Arasṭu Jāh (d.1804), the prime minister of erstwhile Niẓām of Hyderabad State. The translator gives this information in the opening folio (1b) of his other work, *Zīj-i Ṣafdarī.* This unique manuscript is extant in the Salar Jung Museum Library (Hyderabad).[27] This *Zīj* was composed in 1819 and is a translation of *Zīj-i Grah Chandarkāyī Hindī*, which may be identified with the *Grahacandrika Gaṇita* (Calculations for Planets and Moon) by Appaya s/o Marla Perubhaṭṭa (ca. 1491).[28] I have not been able to locate this Persian translation of *Siddhāntaśiromanī* to-date.

6 *Karaṇakutūhala*

1. The *Karaṇakutūhala* is a handbook for the calculation of planetary motion. It was composed by Bhāskara in 1183 [KaKu2008]. Fortunately I have found its complete manuscript in the Raza Library (Rampur) [CPM1996, p. 336]. The title of the Ms is wrongly given as *Risālah dar Hay'at-i Hindī* (Tract on the Indian Astronomy), although it signifies correctly the contents. The folio numbers are also wrong. The correct ones are ff. 118–143. The anonymous scribe has written on f.118a , lines 1–3, clearly as follows.

[25] Cf. [CPM1996, p. 355].

[26] Ms No. Riyāḍī 182, 93 ff; [Sto1972, p. 5] and [Rah1982, p. 392].

[27] See [Ash1988].

[28] For details see [Ans2009, pp. 261–262].

> It is a short [*Risālah*] to learn calculations for the ephemerides of planets, consisting of [41] chapters, which have been translated from *Hindavī* language into Persian, and which Indians call as *Karanktūhal,* [and] was authored by Bhāskarāchārj…

The title of the *Risālah* and the author's name has appeared several times in the text. Note that it had been customary in Indo-Persian languages during the medieval period to call Sanskrit, the language of India (*Hind* in Persian), as *Hindavī.* On the last page f.143a, the last few lines of the colophon carrying the scribe's name, date and place of the writing of this manuscript are missing.

2. An anonymous Persian manuscript is extant in the collection of Punjab University Library (Lahore).[29] It is also in a codex, with ff. 26b–95a. The cataloguer Monzavi presumes the title as *Karanktūhal.* He gives also the text of the beginning of this manuscript, which tallies exactly with that given in the Rampur Ms on f.118. In fact, the few lines given at the end of f.143a of the Rampur Ms are also identical with the colophon of this Lahore Ms, as communicated to me by the former assistant librarian of PUL, S. Jamil Ahmad Rizvi [Riz1980, p. 40]. Therefore I identify the anonymous Lahore manuscript as another copy of the *Karaṇakutūhala.* It may be mentioned that neither Storey nor Pingree mention these Persian translations [Sto1972, pp. 4–5] and [CESS1981, pp. 35–36].
3. Finally, I may add also that a commentary in Persian on the *Karaṇakutūhala* is also extant. Its title is *Sharḥ Frankūhal(a),* comprising 159 ff, in the Punjab Public Library, Lahore (Pakistan) [Abb1963, pp. 270–271]. The cataloguer Abbasi points out on p. 271, that 'the word *Frankūhal* is derived from the Sanskrit word *Bekaran Katūhal*'. The anonymous author of the commentary states in the text that it was written in 1809 Vikrama/ 1751 CE. Actually, this work comprises a translation of the Sanskrit text of the *Karaṇakutūhala* along with a commentary. Since no other Persian commentary is known to-date, this Ms appears to be very important.

7 Conclusion

From the above-mentioned, it is clear that both Muslim and Hindu scholars of medieval India were translating into Persian scientific texts in Sanskrit,

[29] Cf. [Mon1983, p. 363]. The Ms No. is sh/3/102/6261. The scribe is Gul Muḥammad who has not given the date of his writing.

thereby utilizing their endowed receptivity in the best tradition of the West and Central Asia.

Although that trend started in the zenith of Mughal sovereignty during Akbar's time, it continued on its own momentum even during the reigns of Muḥammad Shāh[30] and even of the last Mughal Emperor Bahādur Shāh. That translation activity was not confined evidently to scientific texts, but extended also to Sanskrit epic poems: the *Mahābhārata* and the *Rāmāyaṇa*, the *Upanishads* and literary works as well.[31] I have discussed also in some detail the seven Indo–Persian Zījes compiled in the nineteenth century which are based on the following Sanskrit astronomical works: *Siddhāntaśiromaṇī,* Gaṇeśa Daivajña's *Grahalāghava, Tithicintāmaṇī* and *Laghucintāmaṇī*, the *Brahmatul(ya), Makarandasāraṇī, Grahacandrikā*, and *Bhāsvatī* by Śatānanda of Pūrī (fl. 1099) [Ans2009, pp. 256–263]. On the other hand, Emperors Akbar and Shāh jahān commissioned teams of Muslim and Hindu astronomers to translate into Sanskrit the *Zīj-i Ulugh Beg* (ZUB) and the *Zīj-i Shāhjahānī* (ZShJ). Therefore the following moot questions may be asked to understand fully the medieval Indian scientific scenario.

1. What has been the effect of these translations on Sanskrit scholars and their writings during the late Mughal centuries? Did Islamic astronomy make some dent in Siddhāntic astronomy?
2. What happened to the school of translation founded by Sawa'i Jai Singh after his death? Did the translations in Sanskrit of Euclid's *Elements* and Ptolemy's *Almagest* by Jagannātha find an echo in Sanskrit scientific literature and contributed to the methodological development of *Siddhāntic* astronomy in the late Mughal centuries?

Let me cite here a couple of instances to answer the earlier question in affirmative. Pandit Nityānanda, the court astronomer of Shāhjahān, was ordered to translate in 1630 the ZShJ into Sanskrit. Manuscript copies of this Sanskrit text with the title *Siddhāntasindhu* are extant: One in the Maharajah Man Singh Museum, and three copies in the Royal collection (*Khāṣmohor*) in Jaipur.[32] Nityānanda wrote another astronomical treatise, *Sarvasiddhāntarāja* in 1639, in which Islamic astronomy is presented in the style of Yuga astron-

[30] The copy of the Persian *Bījganit* by ʻAṭāullāh extant in Saʻīdiyah Library (Hyderabad) is dated 1733-34; the reign of Muḥammad Shāh is specifically mentioned by the scribe in the colophon.

[31] Cf. the recent catalogue by [Qas2014] for Sanskrit texts in general. I have already mentioned Fayḍī's translation of the romantic story of emperor Nala and princess Damayanti (*Nal wa Daman* in Persian) as related in the *Mahābhārata.*

[32] One Ms No. 256 (Khāṣmohor collection) bears the seal of Shāhjahān, cf. [Pin1996, pp. 471–48, esp. 476–480].

omy.[33] Further in one Sanskrit Ms No. 5484, Khāṣmohor collection (Jaipur), the lunar theory of the ZUB is compared with that of the ZShJ.

May I finally come to the end by appealing to young generation of scholars to survey Sanskrit *primary* sources thoroughly and to trace this side of the history of science in medieval India, in order to complete the full circle of receptivity by both Persian-knowing scholars and Sanskrit *vidvāns.*

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[33] For details, see [Pin2003, pp. 269–284].

Appendix

List of acronyms for manuscript libraries or collections:

Acronym	Name/place of the repository
ASL	Asiatic Society Library, Calcutta (modern Kolkata).
BHU	Banaras Hindu University, Banaras (modern Varanasi).
BM	British Museum, London.
HL	Hamidia Library, Bhopal.
IAU	Idārah Adabiyāt-i Urdu (Institute of Urdu literature), Hyderabad.
IO	India Office Library, London.
IL	Iqbal Library, Kashmir University, Srinagar, Kashmir.
JHL	Central Library of Jamia Hamdard, New Delhi.
KhB	Khuda Bakhsh Oriental Public Library, Bankipur, Patna.
MAL/coll.	Maulana Azad Library, Aligarh Muslim University, Aligarh, Manuscript collections: Subḥānullāh (Subh), Ḥabīb Ganj (HG), Suleimān (Sul), Quṭbuddin (QD).
MDU	Madrasa Dārul ʻUlūm, Deoband.
MNU	Madarasa Nadwatul ʻUlamā', Lucknow.
MW	Madrasatul Wāʻiẓīn, Lucknow.
NM	National Museum, Karachi, Pakistan.
OUL	Osmania University Library, Hyderabad.
PL	Public Library, Patiala.
PPL	Punjab Public Library, Lahore.
PUL	Punjab University Library, Lahore.
RL	Raza Library, Rampur.
RML	Library of the Raja of Maḥmūdābād, Lucknow.
SCL	State Central Library, formerly Aṣafiyah, Hyderabad. The Mss are now housed in A.P. Govt. Oriental Manuscripts Library and Research Institute, Hyderabad.
SJM	Salar Jung Museum Library, Hyderabad.
SL	Saʻīdyah Library, Hyderabad.
TL	Library of the Arabic and Persian Research Institute, Tonk, Rajasthan.

Note: Published catalogues of almost all these libraries are available. A number of catalogues of Persian manuscripts have been published by the Research Centre of Indo-Persian Language at the Iran Cultural House, New Delhi. However in a couple of madrasas, I found only an accession register. For want of space, I am not listing them in the bibliography. Moreover, I am including in this list the libraries of the India Office and the British Museum (London), since most of their manuscripts belonged originally to the Indian subcontinent. They are now housed in the British Library (London). For Mss collections of other libraries in Berlin, Paris, Vienna, Istanbul and Central Asian countries etc., see the catalogue by Rosenfeld and İhsanoğlu [RE2003].

The use of *Bhūtasaṅkhyās* in the *Līlāvatī* of Bhāskarācārya

Medha Shrikant Limaye*

1 Introduction

The *Bhūtasaṅkhyā* system is a method of expressing numerals with specific words in Sanskrit. In this system words having symbolic meaning are used to connote numerical values. The use of *Bhūtasaṅkhyās* can be traced to several centuries before the common era, as we find its usages in Piṅgala's *Chandaśśāstra*. Sanskrit astronomical and mathematical texts were usually composed in verse form so there was difficulty in denoting very big numbers. Hence this method of object (or word) numerals became popular among Indian astronomers and mathematicians. The system has also been used in inscriptions and manuscripts for specifying dates. Various innovative methods of using *Bhūtasaṅkhyās* are found in mathematical texts. Bhāskarācārya was a skilled writer and one of the striking features of the *Līlāvatī* is the choice of precise and specific words. This paper throws light on the *Bhūtasaṅkhyās* used in the *Līlāvatī* of Bhāskarācārya.

2 Numerical notations in Sanskrit

Numbers have been an important medium of counting since primitive times and the base of the number system was ten in many early civilizations. In ancient India too the first nine natural numbers, ten and several powers of ten were denoted by words. Sanskrit words for the first nine natural numbers

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are *eka, dvi, tri, catur, pañca, ṣaṭ, sapta, aṣṭa*, and *nava*. The words for groups of ten are found in the *Ṛgveda*.[1] The system of naming the notational places increasing successively in powers of ten was also developed at that time. In the *Taittirīya Saṁhitā* the list is found up to 12$^{\text{th}}$ power of ten and the names are *eka, daśa, śata, sahasra, ayuta, niyuta, prayuta, arbuda, nyarbuda, samudra, madhya, anta* and *parārdha*.[2]

3 The continuity of tradition

The continuity of this tradition is seen in the works of later mathematicians. Āryabhaṭa names ten notational places.[3] Śrīdharācārya,[4] Śrīpati[5] and Bhāskarācārya[6] enumerate eighteen numeral denominations.

Mahāvīrācārya names twenty-four places.[7] As regards to this, Alberuni points out [AIB1910, p. 174]:

[1] आ द्वाभ्यां हरिभ्यामिन्द्र याह्या चतुर्भिरा षड्भिर्हूयमानः।
आष्टाभिर्दशभिः सोमपेयमयं सुतः सुमख मा मृधस्कः॥
आ विंशत्या त्रिंशता याह्यर्वाङा चत्वारिंशता हरिभिर्युजानः।
आ पञ्चाशता सुरथेभिरिन्द्रा षष्ट्या सप्तत्या सोमपेयम्॥
आशीत्या नवत्या याह्यर्वाङा शतेन हारिभिरुह्यमानः।
अयं हि ते शुनहोत्रेषु सोम इन्द्र त्वाया परिषिक्तो मदाय॥ [RgSa1936, 2.18.4–6]

[2] इमा मे अग्न इष्टका धेनवः सन्त्वेका च शतं च सहस्रं चायुतं च नियुतं च प्रयुतं चार्बुदं च न्यर्बुदं च समुद्रश्च मध्यं चान्तश्च परार्धश्चेमा मे अग्न इष्टका धेनवः सन्तु । [TSaṃ1990, 4.4.11]

[3] एकं दश च शतं च सहस्रं त्वयुतनियुते तथा प्रयुतम्।
कोट्यर्बुदं च वृन्दं स्थानात् स्थानं दशगुणं स्यात्॥ [AB1976, v. 2.2]

[4] एकं दश शतमस्मात्सहस्रमयुतं ततः परं लक्षम्। प्रयुतं कोटिमथार्बुदमब्जं खर्वं निखर्वं च॥
तस्मान्महासरोजं शंकुं सरितां पतिं ततस्त्वन्त्यम्। मध्यं परार्द्धमाहुर्यथोत्तरं दशगुणाः तज्ज्ञाः॥ [PāGa1959, vv. 7–8]

[5] एकं दशस्थानमथो शतं च सहस्रमस्मादयुतं च लक्षम्।
अनन्तरं तु प्रयुतं च कोटिरथार्बुदं पद्ममतश्च खर्वम्॥
निखर्वसंज्ञं च महासरोजं शङ्कुः समुद्रोऽन्त्यमतश्च मध्यम्।
परार्धमित्याहुरिमां हि सङ्ख्यां यथोत्तरं स्थानविदो दशघ्नीम्॥ [GaTi1937, vv. 2–3]

[6] एकदशशतसहस्रायुतलक्षप्रयुतकोटयः क्रमशः।
अर्बुदमब्जं खर्वनिखर्वमहापद्मशङ्कवस्तस्मात्॥
जलधिश्चान्त्यं मध्यं परार्धमिति दशगुणोत्तराः संज्ञाः।
संख्यायाः स्थानानां व्यवहारार्थं कृताः पूर्वैः॥ [Līlā1937, vv. 10–11]

[7] एकं तु प्रथमस्थानं द्वितीयं दशसंज्ञिकम् । तृतीयं शतमित्याहुः चतुर्थं तु सहस्रकम्॥
पञ्चमं दशसाहस्रं षष्ठं स्याल्लक्षमेव च । सप्तमं दशलक्षं तु अष्टमं कोटिरुच्यते॥
नवमं दशकोट्यस्तु दशमं शतकोटयः । अर्बुदं रुद्रसंयुक्तं न्युर्बुदं द्वादशं भवेत्॥
खर्वं त्रयोदशस्थानं महाखर्वं चतुर्दशम् । पद्मं पञ्चदशं चैव महापद्मं तु षोडशम्॥

> I have studied the names of orders in various languages with all kinds of people with whom I have been in contact, and have found that no nation goes beyond thousand. The Arabs too stopped with the thousand which is certainly the most correct and the most natural thing to do.…Those, however who go beyond the thousand in their numerical system are the Hindus. …They extend the names of the *orders* of the numbers until the 18th order for religious reasons, the mathematicians being assisted by the grammarians with all kinds of etymologies.

4 The *Bhūtasaṅkhyā* system

In spite of these developments, there was difficulty in denoting very big numbers in astronomical and mathematical texts. To overcome this difficulty, a novel system using the alphabet was used by Āryabhaṭa to denote numbers. But most of the letter chronograms, formed according to his system, were very difficult to pronounce and remember. Probably for this reason, the system was not accepted by later mathematicians. Hence another method was put to use to represent numbers, namely, the *Bhūtasaṅkhyā* system. It is a method of expressing numerals with specific words having symbolic meaning in Sanskrit. In this system numbers are expressed by names of things or concepts which connote numbers. Varāhamihira, Brahmagupta, Bhāskara I, Mahāvīrācārya and Śrīpati used this system extensively in their texts and Bhāskarācārya followed this tradition. This paper investigates how Bhāskarācārya has made use of the *Bhūtasaṅkhyās* (object numerals) in the *Līlāvatī*.

5 Historical background of the *Bhūtasaṅkhyā* system

In the *Kātyāyana Śrautasūtra* the names of the metres *Gāyatrī* and *Jagatī* have been used to denote the numbers twenty-four and forty-eight respectively. But early usage of this system was without place-value. It was used with place-value notation in astronomical texts like the *Sūryasiddhānta* and the *Pañcasiddhāntikā* as well as in the *Agni-Purāṇa* [DA2001, pp. 53–63].

क्षोणी सप्तदशं चैव महाक्षोणी दशाष्टकम् । शङ्खं नवदशं स्थानं महाशङ्खं तु विंशकम्॥
क्षित्यैकविंशतिस्थानं महाक्षित्या द्विविंशकम् । त्रिविंशकमथ क्षोभं महाक्षोभं चतुर्नयम्॥
[GSS1912, vv. 63–68]

In the *Vedāṅga Jyotiṣa*, the words *rūpa* and *bhasamūha* have been used to represent numerals 1 and 27 respectively.[8] In metrics, while giving the definitions of metres, the caesura was given by using word numerals. For example, in the definition of the metre *mandākrāntā*, the words *ambudhi, rasa* and *naga* are used to denote 4, 6 and 7 respectively.[9] The system is used in the dates of inscriptions and in manuscripts too.

6 Advantages of the *Bhūtasaṅkhyā* system in mathematical texts

This system of metonymic expression of numbers was quite suitable for number-centric disciplines like mathematics and astronomy. Most of the Sanskrit texts were composed in verse form and a common feature of mathematical works was the use of various suitable metres. The metrical verse form was close to music and it helped a lot for memorization through recitation. A great variety of synonymous words in Sanskrit containing different numbers of syllables referring to a particular object could be employed to denote a particular number by using the *Bhūtasaṅkhyā* system.

Although metrical convenience was the prime reason behind developing this system, it served another purpose of bringing together mathematics and culture. Various familiar concepts from the *vedas, purāṇas*, mythology and environment were utilized for object numerals. The use of object numerals indirectly helped to enrich pupils with our history and culture while learning mathematics. Moreover the *Bhūtasaṅkhyā* system had a mnemonic value as it is always easier to remember numerical data with object-names. In addition, it helped in conserving the texts with better accuracy. Regarding this facet P. V. Kane says [Kan1958, pp. 701–703]:

> This was a very reliable method when in astronomy huge figures had to be employed and works were not printed but only copied by hand. In ancient times the writers of manuscripts might often omit zeros or other figures, but if words with a fixed meaning in relation to numbers were used, they would not be so easily dropped and as many works were metrical, the omission of a word, if any, might have been far more easily detected.

[8] विषुवं तद्गुणं द्वाभ्यां रूपहीनं तु षड्गुणम्। यल्लब्धं तानि पर्वाणि त(थार्ध) सा तिथिर्भवेत्॥ *Āj. 31*
विषुवन्तं द्विरभ्यस्तं रूपोनं षड्गुणीकृतम्। पक्षा यदर्धं पक्षाणां तिथिः स विषुवान् स्मृतः॥ *Yj. 23*
तिथिमेकादशाभ्यस्तां पर्वभांशसमन्विताम्। विभज्य भसमूहेन तिथिनक्षत्रमादिशेत्॥ *Yj. 20*
[VJ1985, pp. 25 and 29]

[9] मन्दाक्रान्ताम्बुधिरसनगैर्मोभनौ तौ गयुग्मम्। [Dev1997, p. 1092]

7 The use of *Bhūtasaṅkhyā* System in the *Līlāvatī*

It is observed that various innovative methods of using *Bhūtasaṅkhyās* are found in mathematical and astronomical texts. Single words indicating a digit were commonly used but sometimes a single word denoted a two-digit number too. Appropriate words were placed one after the other with place value to form long numbers and read from right to left as per the principle – *aṅkānām vāmato gatiḥ* (the numerals proceed to the left). Sometimes a compound of object numerals and usual numerals was conveniently used. Bhāskarācārya has employed all these varieties in the *Līlāvatī*.

The following classification of *Bhūtasaṅkhyās* in the *Līlāvatī* will give the idea about the variety of words used by Bhāskarācārya to denote numbers.

Bhūtasaṅkhyās related to astronomy and the calendar

Kha, Nabhas, Abhra: The Sky (0): Although *śūnya* is frequently used in the *Līlāvatī* to denote zero, the synonyms of the sky *kha, nabhas, abhra* are used to denote zero because the sky was considered to be an empty space.

Indu, Candra: The Moon (1): The moon is the only natural satellite of the earth and an object giving light at night.

Ku, Bhū : The Earth (1): The earth is unique in many respects.

Yuga: Age or Era (4): According to the Hindu calendar, *yuga* is the name of an epoch or an era. These epochs or long periods of years are four in number. *Viṣṇupurāṇa*[10] mentions their names as *Kṛta* or *Satya, Tretā, Dvāpara*, and *Kali*.

Go: Planets (9): One of the meanings of the word *go* is 'a region of the sky' [Mon1979]. In Hindu astrology, nine planets were known as *Maṅgala, Budha, Bṛhaspati, Śukra, Śani, Rāhu, Ketu, Sūrya* and *Candra*.

Dik: The Directions (10): The numbers eight and ten were usually denoted by the word *dik*. The Sanskrit names for the eight cardinal points of the universe are *pūrva, paścima, uttara, dakṣiṇa, āgneya, nairṛtya, vāyavya, īśānya*. The words *ūrdhva* and *adhara* are for the upward and downward directions respectively. By adding these two directions, the number becomes ten. In the *Līlāvatī, dik* denotes ten.

[10] कृतं त्रेता द्वापरश्च कलिश्चैव चतुर्युगम्। प्रोच्यते तत्सहस्रं च ब्रह्मणो दिवसं मुने॥
चत्वारि भारते वर्षे युगान्यत्र महामुने। कृतं त्रेता द्वापरश्च कलिश्चान्यत्र न क्वचित्॥
[ViPu1986, 701–703]

Tithi: Lunar days (15): In the Hindu calendar, the basic unit is *tithi* or lunar day. There are fifteen days in one *pakṣa*.

Bha: Lunar mansions (27): In Hindu astronomy, twenty-seven lunar mansions – *nakṣatras* —are known. So the word *bha* meaning *nakṣatras* occurs for twenty-seven.

Bhūtasaṅkhyās* related to deities and concepts mentioned in *Vedas* and *Purāṇas

Yama: A pair (2): Yama, the ruler of the departed, is mentioned by the word *mithuna* (pair) in the *Ṛgveda*.[11]

He is paired with a twin sister Yamī or Yamunā.[12]

Agni: Fire (3): The sacrificial fire was of three kinds namely *gārhapatya, āhavanīya* and *dakṣiṇa*.[13]

Rāma: (3): Traditionally three great persons are named as Rāma namely, Paraśurāma, Rāmacandra and Balarāma. Paraśurāma was the son of Jamadagni.[14] Rāmacandra, was the hero of the epic *Rāmāyaṇa* and Balarāma was the elder brother of Kṛṣṇa, represented as armed with a plough-share and known as the patron of agriculture.[15] Hence the word is used to denote three.

Sāgara, Abdhi, Ambhodhi: Ocean (4): This is a symbolic expression for the number four. Four principal oceans were reckoned, one corresponding to every quarter of the sky [Mon1979].

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[11] यमा चिदत्र यमसूरसूत जिह्वाया अग्रं पतदा ह्यस्थात्
वपूंषि जाता मिथुना सचेते तमोहना तपुषो बुध्न एता॥ [ṚgSa1936, 3.39.3]

[12] धर्मराजः पितृपतिः समवर्ती परेतराट्।
कृतान्तो यमुनाभ्राता शमनो यमराड्यमः॥|
[AmKo2012, 1.1.61]

[13] दक्षिणाग्निर्गार्हपत्याहवनीयौ त्रयोऽग्नयः।
अग्नित्रयमिदं त्रेता प्रणीतः संस्कृतोऽनलः॥
[AmKo2012, 2.6.13–14]

[14] जमदग्नेऽस्तु चत्वार आसन्पुत्रा महात्मनः।
रामस्तेषां जघन्योऽभूदजघन्यैर्गुणैर्युतः॥
[MaBt1929, 1.66.48]

[15] बलभद्र प्रलम्भघ्नो बलदेवोऽच्युताग्रजः।
रेवतीरमणो रामः कामपालो हलायुधः॥
[AmKo2012, 1.1.24]

Veda: (4): The word denotes four as the *Vedas* are four namely *Ṛgveda, Yajurveda, Sāmaveda* and *Atharvaveda.*

Iṣu, Bāṇa, Śara: Arrow (5): *Madana* or *Kāmadeva*, Indian god of love, is traditionally represented as armed with five floral arrows, namely, *Aravinda, Aśoka, Cūta, Navamallikā* and *Nīlotpala.* Also five physical arrows of Kāmadeva are mentioned as *unmādana, tāpana, śoṣaṇa, stambhana*, and *sammohana.* [16]

Aśva, Turaga: Horse (7): The *Ṛgveda Saṁhitā*, mentions that the chariot of the Sun has seven horses.[17]

Adri, Śaila: Mountain (7): According to the *Viṣṇupurāṇa*, there were seven principal mountains called *kulācalas* namely *Mahendra, Malaya, Sahya, Śuktimat, Ṛkṣa, Vindhya* and *Pāripātra.*[18]

Kumbhin: Elephant (8): Eight elephants, *diggajas*, are associated with eight gods known as *aṣṭadikpālas*, who guard the eight cardinal directions. The names of the elephants are *Airāvata, Puṇḍarīka, Vāmana, Kumuda, Añjana, Puṣpadanta, Sārvabhauma* and *Supratīka.*[19]

Vasu: (8): *Vasus* form a group of eight deities known as elemental gods representing aspects of nature. According to the *Mahābhārata* the group of them includes *Āpa, Dhruva, Soma, Dhara, Anila, Anala, Pratyūṣa* and *Prabhasa.*[20]

Rudra, Madanāri, Īśa: (11): Synonyms of Śiva represent the numeral eleven as it was believed that Śiva has eleven forms of *Rudras*. The *Mahābhārata* gives the names as *Mṛgavyādha, Sarpa, Nirṛti, Ajaikapādat, Ahirbudhnya, Pinākin, Dahana, Īśvara, Kapāli, Sthāṇu* and *Bhaga.*[21]

Sūrya, Arka, Tigmakara, Divākara, Ravi: Sun-God (12): The concept of twelve suns *(Ādityas)* corresponds to twelve months of a year. According to

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[16] अरविन्दं अशोकं च चूतं च नवमल्लिका। नीलोत्पलं च पञ्चैते पञ्चबाणस्य सायकाः॥
उन्मादनस्तापनश्च शोषणः स्तम्भनस्तथा। सम्मोहनश्च कामस्य पञ्च बाणाः प्रकीर्तिताः॥
[AmKo2012, 1.1.57, vv. 1–2]

[17] सप्त त्वा हरितो रथे वहन्ति देव सूर्य शोचिष्केशं विचक्षण। [ṚgSa1936, 1.50.8]

[18] महेन्द्रो मलयः सह्यः शुक्तिमान् ऋक्षपर्वतः। विन्ध्यश्च पारिपात्रश्च सप्तात्र कुलपर्वताः॥|
[ViPu1986, 2.3.3]

[19] ऐरावतः पुण्डरीको वामनः कुमुदोञ्जनः। पुष्पदन्तः सार्वभौमः सुप्रतीकश्च दिग्गजाः॥
[ViPu1986, 1.3. 182–183][AmKo2012, 1.3.3. 182–183]

[20] धरो ध्रुवश्च सोमश्च अहश्चैवानिलोऽनलः। प्रत्यूषश्च प्रभासश्च वसवोष्टौ प्रकीर्तिताः॥
[MaBt1929, 1.66.18]

[21] ब्रह्मणो मानसाः पुत्रा विदिताः षण्महर्षयः। एकादश सुताः स्थाणोः ख्याताः परमतेजसः॥
मृगव्याधश्च सर्पश्च निर्ऋतिश्च महायशाः। अजैकपादहिर्बुध्न्यः पिनाकी च परंतपः॥
दहनोऽथेश्वरश्चैव कपाली च महाद्युतिः। स्थाणुर्भगश्च भगवान् रुद्रा एकादश स्मृताः॥
[MaBt1929, 1.66.1–3]

the *Mahābhārata* their names are *Dhātā, Mitra, Aryaman, Śakra, Varuṇa, Aṁśa, Bhaga, Vivasvān, Pūṣan, Savitṛ, Tvaṣṭṛ* and *Viṣṇu.*[22]

Viśva: (13): *Viśvedevas* denote a group of deities. In the *Ṛgveda*, a number of hymns are addressed to them. Although the word *Viśva* is found for the number thirteen, reference to thirteen deities in the group is not found in *Ṛgveda*. However, according to the *Viṣṇupurāṇa* they were the ten sons of *Viśvā*, daughter of *Dakṣa.*[23] According to Monier Williams, their names are, *Vasu, Satya, Kratu, Dakṣa, Kāla, Kāma, Dhṛti, Kuru, Purūravas, Mādravas* and two others are added by some viz. *Rocaka* or *Locana* and *Dhvani* [or *Dhūri*, or this may make thirteen [Mon1979].

Indra, Manu, Śakra: (14): In the *Purāṇas* it is said that fourteen *manvantaras* make a *kalpa* - a period corresponding to a day of Brahmā and every *Manvantara* has one *Indra.* According to the *Viṣṇupurāṇa*, their names are *Svāyambhuva, Svārociṣa, Uttama, Tāmasa, Raivata, Cākṣuṣa, Vaivasvata, Sāvarṇi, Daśasāvarṇi, Brahmasāvarṇi, Dharmasāvarṇi, Rudraputra, Rauncya* and *Bhautya.*[24]

Bhūtasaṅkhyās related to philosophy

Guṇa : Quality (3): According to Hindu philosophy, *guṇas*, meaning qualities or fundamental operating principals of *prakṛti* are three in number. They are *sattva, rajas* and *tamas.*[25]

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[22] अदित्यां द्वादशादित्याः संभूता भुवनेश्वराः। ये राजन्नामतस्तांस्ते कीर्तयिष्यामि भारत॥
धाता मित्रोर्यमा शक्रो वरुणस्त्वंश एव च। भगो विवस्वान् पूषा च सविता दशमस्तथा॥
एकादशस्तथा त्वष्टा द्वादशो विष्णुरुच्यते। जघन्यजस्तु सर्वेषामादित्यानां गुणाधिकः॥
[MaBt1929, 1.65.14–16]

[23] अरुन्धती वसुर्यामी लम्बा भानुर्मरुत्वती। सङ्कल्पा च मुहूर्ता च साध्या विश्वा च ता दश॥
धर्मपत्न्यो दश त्वेता: तदपत्यानि मे श्रृणु। विश्वेदेवास्तु विश्वायाः साध्या साध्यात् व्यजायत॥
[ViPu1986, 1.15.105–106]

[24] स्वायंभुवो मनुः पूर्वो मनुः स्वारोचिषस्तथा। उत्तमस्तामसश्चैव रेवतश्चाक्षुषस्तथा॥
षडेते मनवोऽतीताः साम्प्रतन्तु रवेः सुतः। वैवस्वतोऽयं यस्यैतत् सप्तमं वर्ततेऽन्तरम्॥
सावर्णिस्तु मनुर्योऽसौ मैत्रेय भविता ततः। सुतपाश्चामिताभाश्च मुख्याश्चापि तदा सुराः॥
नवमो दशसावर्णो मैत्रेय भविता मनुः। पारा मरीचिगर्भाश्च सुधर्माणस्तथा त्रिधा॥
दशमो ब्रह्मसावार्णिर्भविष्यति मुने मनुः। सुधामानो विशुद्धाश्च शतसंख्यास्तथा सुराः॥
एकादशश्च भविता धर्मसावर्णिको मनुः। विहङ्गमाः कामगमा निर्वाणरतयस्तथा॥
रुद्रपुत्रस्तु सावर्णो भविता द्वादशो मनुः। ऋतधामा च तत्रेन्द्रो भविता श्रृणु मे सुरान्॥
त्रयोदशो रौच्यनामा भविष्यति मुने मनुः। सुत्रामाणः सुधर्माणः सुकर्माणः तथापराः॥
भौत्यश्चतुर्दशश्चात्र मैत्रेय भविता मनुः। शुचिरिन्द्रः सुरगणास्तत्र पञ्च श्रृणुष्व तान्॥
[ViPu1986, 3.1.6–7, 3.2.15, 20, 24, 28, 32, 36, 40]

[25] त्रिगुणं व्यक्तं सत्त्वरजस्तमांसि त्रयो गुणा यस्येति। [SaKa1837, *Kārikā Bhāṣya* 11, p. 10]

Bhūta: Elements (5): In *Sāṅkhya* philosophy the traditional number of gross elements was five and they were called *pañcamahābhūtas*. They are *Pṛthvī* (earth), *Jala* (water), *Agni* (fire), *Vāyu* (air) and *Ākāśa* (sky).[26] It was believed that the whole universe is pervaded by these elements. So the word *bhūta* is used to denote five.

Jina: (24): According to Jainism, *Jina* is the name which implies saints. *Jina* is especially one venerated as a *tīrthaṅkara*. Names of twenty-four *tīrthaṅkaras* are frequently referred to in Jaina texts. According to a treatise *Devatāmūrtiprakaraṇam*, their names are *Ṛṣabha, Ajita, Sambhava, Abhinandana, Sumati, Padmaprabha, Supārśva, Candraprabha, Puṣpadanta, Śītala, Śreyāṁsa, Vāsapatnya, Vimala, Ananta, Dharma, Śānti, Kunthunātha, Ara, Malli, Munisuvrata, Nami, Nemi, Pārśvanātha, Vardhamāna*.[27] *Svayambhūstotra* composed by Ācārya Samantabhadra contains an adoration of these twenty-four *tīrthaṅkaras* [SvS2015].

Tattva: (25): According to *Sāṅkhya* philosophy, there are twenty-five elements or *tattvas*. They are *avyakta, buddhi, ahaṅkāra*, the five *tan-mantras* or subtle elements, the five *mahābhūtas* or gross elements, the eleven organs including five *jñānendriyāṇi*, five *karmendriyāṇi* and *manas* or mind and the last one *puruṣa*, in all making the number twenty-five.[28]

विशेषः कालिकोऽवस्था गुणाः सत्त्वं रजस्तमाः॥

[AmKo2012, 1.4.29]

[26] किं च पञ्चभ्यः पञ्चभूतानि तस्माद्षोडशकारणात् पञ्चभ्यस्तन्मात्रेभ्यः सकाशात् पञ्च वै महाभूतान्युत्पद्यन्ते। यदुक्तं शब्दतन्मात्रादाकाशं स्पर्शतन्मात्राद्वायुः रूपतन्मात्रात्तेजः रसतन्मात्रादापः गन्धतन्मात्रात् पृथिवी एवं पञ्चभ्यः परमाणुभ्यः पञ्चमहाभूतान्युत्पद्यन्ते।

[SaKa1837, *Kārikā Bhāṣya* 22, p. 20]

[27] ऋषभश्चाजितश्चैव सम्भवश्चाभिनन्दनः। सुमतिः पद्मप्रभश्च सुपार्श्वः सप्तमानतः॥
चन्द्रप्रभः पुष्पदन्तः शीतलो दशमो मतः। श्रेयांसो वासपत्यश्च विमलोऽनन्तधर्मकौ॥
शान्तिश्च कुन्थुनाथारमल्लयो मुनिसुव्रतः। नमिर्नेमिः पार्श्वनाथो वर्द्धमानस्ततः परम्॥
चतुर्विंशतिरित्येके मतास्तीर्थंकरा बुधैः। [DeMu2003, 7.1–3]

[28] पञ्चविंशतितत्त्वज्ञो यत्र तत्राश्रमे वसेत्
जटी मुण्डी शिखी वापि मुच्यते नात्र संशयः।

[SaKa1837, *Kārikā Bhāṣya* 1, p. 1]

व्यक्ताव्यक्तज्ञविज्ञानात् तत्र व्यक्तं महदादिबुद्धिरहङ्कारः पञ्च तन्मात्राणि एकादशेन्द्रियाणि पञ्चमहाभूतानि। अव्यक्तं प्रधानं। ज्ञः पुरुषः। एवमेतानि पञ्चविंशति तत्त्वानि व्यक्ताव्यक्तज्ञानि कथ्यन्ते एतद्विज्ञानाच्छ्रेय इत्युक्तं च पञ्चविंशतितत्त्वज्ञ इति। [SaKa1837, *Kārikā Bhāṣya* 2, p. 3]

प्रकृतिः पुरुषो बुद्धिरहंकारः पञ्चतन्मात्रा एकादशेन्द्रियाणि पञ्चमहाभूतानि इत्येतानि पञ्चविंशति तत्त्वानि।

[SaKa1837, *Kārikā Bhāṣya* 22, p. 21]

Other concepts used as *Bhūtasaṅkhyās*

Rūpa: Form, Figure (1): The word is used as a term for number one meaning a single specimen or exemplar.

Netra, Locana: Eyes (2): As every human being has two eyes, the words *netra, locana* denote two.

Yuga, Yugala, Yugma: A pair (2): This word, literally meaning a yoke, is used to denote two.

Rasa: Taste (6): This word is used for six because according to *Āyurvedic* tradition, there are six types of taste. *Caraka Saṁhitā* mentions them as *madhura, amla, lavaṇa, kaṭu, tikta, kaṣāya.*[29] Incidentally, Bhāskarācārya has listed all these *ṣaḍrasas* in a numerical problem on combinations.[30]

Svara: Notes (7): In music seven principal notes of the scale are found. The *Nāṭyaśāstra* of Bharatamuni mentions them as *ṣaḍja, ṛṣabha, gāndhāra, madhyama, pañcama, dhaivata, niṣāda.*[31]

Aṅka: Digits (9): There are nine digits from one to nine.

Nanda: (9): It is the name of a dynasty of kings in Magadha. There were nine kings who ruled the kingdom.

Nṛpa: King (16): Sixteen great kingdoms existed from 600 BCE to 400 BCE in India. According to the ancient Buddhist text *Aṅguttara Nikāya* those *mahājanapadas* were *Aṅga, Magadha, Kāśī, Kosala, Vajji, Malla, Ceti, Vaṁsa, Kuru, Pañcāla, Maccha, Sūrasena, Assaka, Avantī, Gāndhāra* and *Kamboja* [AṅNi1899, p. 252]. The word *nṛpa* has reference to the kings of these sixteen kingdoms.

Dhṛti: (18): *Dhṛti* is the name of a metre having eighteen syllables in each *pāda.*

Atidhṛti: (19): *Atidhṛti* is the name of a metre having nineteen syllables in each *pāda.*

Nakha: Nail (20): Humans have twenty nails.

Utkṛti: (26): *Utkṛti* is the name of a metre having twenty-six syllables in each *pāda.*

Danta: Teeth (32): A grown-up human being has thirty-two teeth.

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[29] षडेव रसा इत्युवाच भगवानात्रेयः पुनर्वसुः मधुराम्ललवणकटुतिक्तकषायाः॥
[CaSa1896, ch. 1.26.9]

[30] एकद्वित्र्यादियुक्ता मधुरकटुकषायाम्लकक्षारतिक्तैः।
एकस्मिनषड्रसैः स्युर्गणक कति व्यञ्जने व्यक्तिभेदाः॥ [Līlā1937, v. 116]

[31] षड्जश्च ऋषभश्चैव गान्धारो मध्यमस्तथा। पञ्चमो धैवतश्चैव सप्तमश्च निषादवान्॥
[NāŚā2006, 28.21]

8 Notable features of *Bhūtasaṅkhyās* in the *Līlāvatī*

Bhāskarācārya displays high literary skill in the *Līlāvatī*. Various metrical forms such as *Mālinī, Śārdūlavikrīḍitam, Śikhariṇī* and *Vasantatilakā* have been used by him in the *Līlāvatī*. So object numerals were very suitable for such elegant *akṣara-gaṇa-vṛttas*. Bhāskarācārya had a wide vocabulary so he has used a variety of object numerals related to many branches of culture. He has used some uncommon ones too. For instance, the words *Arka, Divākara, Sūrya, Ravi* used by him are commonly found for the number twelve but the word *Tigmakara*, meaning the sun, is not frequently used in mathematical texts. Similarly the word *Kumbhin* (elephant) for eight and the words *atidhṛti* and *utkṛti* denoting nineteen and twenty-six respectively are rare.

Although Bhāskarācārya has taken full advantage of the *Bhūtasaṅkhyā* system, he has used the system wisely and selectively. He was not only a mathematician but also a good teacher. His book is written following the traditionally important maxim "proceed from the easy to the more difficult". Most of the topics discussed in the chapters ending with *Prakīrṇaka*, are comparatively easier and the numerical examples too are composed in simple language rarely employing object numerals in those topics. The first instance of object numerals on a large scale is found in the chapter on mixtures. There, while giving numerical problems based on the formulae to find weight and fineness of gold, he has used several object numerals in three numerical problems [Līlā1937, vv. 104, 107, 109].

It is remarkable that all these numerical examples occur in the same mathematical topic and contain similar styles of language. In definitions of measurement units (*Paribhāṣā*), only in the table for measurements of gold does he use the word *Indra* for fourteen [Līlā1937, v. 3]. Also in the chapters on eight operations on numbers, the word *Divākara* is the only word found in the numerical example on multiplication where the multiplier is twelve [Līlā1937, v. 17].

Bhāskarācārya has freely used the *Bhūtasaṅkhyā* system particularly in the chapter on *Kṣetravyavahāra* (Geometry) of the Līlāvatī. A typical instance can be mentioned to illustrate the ease with which he has used the object numerals in a geometrical formula giving lengths of the sides of regular polygons of sides 3, 4, 5, 6, 7, 8 and 9 inscribed in a circle of diameter 120000 units. As the verse contains big numbers consisting of five or six digits, the system was extremely suitable for expressing the numerical data.[32]

[32] त्रिद्व्यङ्काग्निनभश्चन्द्रैस्त्रिबाणाष्टयुगाष्टभिः। वेदाग्निबाणखाश्वैश्च खखाभ्रभ्ररसैः क्रमात्॥
बाणेषुनखबाणैश्च द्विद्विनन्देषुसागरैः। कुरामदशवेदैश्च वृत्तव्याससमाहते॥

Similarly one may find verses employing both the standard word numerals and the object numerals. For instance, in the verse giving the value of π, the well-known ratio of circumference and diameter of the circle, the author has used usual numeral *dvāvimśati* for twenty-two and the object numeral *śaila* for seven in the second line where the gross value of π equal to $\frac{22}{7}$ is given. But in the first line the word *bhanandāgni* denotes the number 3927 where *bha* stands for 27, *nanda* for 9 and *agni* for 3. The word *khabāṇasūrya* denotes the number 1250 where *kha* stands for 0, *bāṇa* for 5 and *sūrya* for 12. So the value is obtained as $\frac{3927}{1250}$ equal to 3.1416, which is a better approximation of π. Here again two object numerals, *bha* and *sūrya* have been used to denote two-digit numbers 27 and 12 respectively.[33]

Occasionally, original number-words are mingled with the *Bhūtasaṅkhyās* [Līlā1937, v. 191]. Here, the word *khāṣṭayama (kha + aṣṭa + yama)* has been used where *kha* denotes zero, *aṣṭa* is usual numeral for eight and *yama* denotes two. In another example Bhāskarācārya uses the word *daśeśārka* to denote three different numbers giving values of the sides of three irregular solids. Here *daśa* denotes ten with usual numeral and *īśa* and *arka* stand for eleven and twelve respectively [Līlā1937, vv. 215–216]. Also the word *ṣaṭsvareṣu* denotes a three digit number 576 where *ṣaṭ* means six, *svara* denotes seven and *iṣu* denotes five [Līlā1937, v. 223].

9 Conclusion

The above analysis makes it clear that although a language tool is at hand, a skilled writer like Bhāskarācārya handles it in a careful and effective manner. Moreover, Bhāskarācārya as an excellent teacher judiciously employs the *Bhūtasaṅkhyās* wherever it was necessary for metrical structure. By selecting *Bhūtasaṅkhyās* from various branches of culture he also made his teaching of mathematics culturally relevant. This was possible for him because he had an excellent command over language.

Culturally relevant teaching has now become more widely known and accepted method to help students excel in education. It is noteworthy that ancient Indian mathematicians were thinking in a similar way and they tried to teach mathematics in the context of different cultural ideas. One of the distinctive features of the *Līlāvatī* is the choice of precise and specific words and

खखखाभ्रर्कसम्भक्ते लभ्यन्ते क्रमशो भुजाः।
वृत्तान्तत्र्यस्रपूर्वाणां नवास्रान्तं पृथक् पृथक् ॥ [Līlā1937, vv. 206–208]

[33] व्यासे भनन्दाग्निहते विभक्ते खबाणसूर्यैः परिधिः स सूक्ष्मः।
द्वाविंशतिघ्ने विहृतेऽथ शैलैः स्थूलोऽथवा स्याद्व्यवहारयोग्यः॥ [Līlā1937, v. 199]

such clever use of language can be experienced in the use of *Bhūtasaṅkhyās* too. Hence to conclude it can be said that Bhāskarācārya's mathematical skills and literary skills went hand-in-hand in the *Līlāvatī*.

Appendix

The table below presents a list of the *Bhūtasaṅkhyās* used in the *Līlāvatī*.

Word	Number	Verse Numbers	Page Numbers
अग्नि	three	199, 203, 206	197-198, 203, 207
अङ्क	nine	170, 206	158, 207
अतिधृति	ninteen	191	190
अद्रि	seven	232	240
अब्धि	four	212	216
अभ्र	zero	206, 208	207
अम्भोधि	four	265	279
अर्क	twelve	86, 104, 164, 207, 208, 211, 215, 235, 240	81, 100, 152, 207, 214, 221, 244, 247-248
अश्व	seven	206	207
इन्दु	one	175, 191	165, 190
इन्द्र	fourteen	3, 109,	8, 103
इषु	five	207, 223, 232	207, 230, 240
ईश	eleven	107, 215	102, 221
उत्कृति	twenty-six	191	190
कु	one	119, 207	113, 207
कुम्भिन्	eight	265	279
ख	zero	45, 46, 47, 191, 199, 206, 208, 211, 255, 267	39, 40, 190, 197, 198, 207, 214, 265-266, 281
गुण	three	109, 191	103, 190
गो	nine	191	190

चन्द्र	nine	109, 206. 233	103, 207, 241-242
जिन	twenty-four	191	190
तत्त्व	twenty-five	191	190
तिग्मकर	twelve	219	225
तिथि	fifteen	80, 167, 191, 230	75, 154, 190, 238
तुरग	seven	202	202
दन्त	thirty-two	226	233
दिक्	ten	104	100
दिवाकर	twelve	17, 170	16, 158
धृति	eighteen	191	190
नख	twenty	175, 207, 224	165, 207, 231
नन्द	nine	199, 207, 233	197-198, 207, 241-242
नभस्	zero	206	207
नृप	sixteen	150, 224	141, 231
नेत्र	two	107	102
बाण	five	199, 206, 207	197-198, 207
भ	twenty-seven	199, 203	197-198, 203
भूत	five	265	279
मदनारि	eleven	175	165
मनु	fourteen	167	154
यम	two	191	190
युग	four	104, 206	100, 207
युग	two	25, 69, 131	24, 65-66, 121
युगल	two	54, 99, 100	49-51, 93, 95
युग्म	two	67	64
रस	six	174, 206, 232	163, 207, 240
रवि	twelve	170	158
राम	three	207	207
रुद्र	eleven	104, 203	100, 203

रूप	one	61, 244, 257, 268	56-57, 253, 267, 282
लोचन	two	104	100
वसु	eight	107	102
विश्व	thirteen	104, 233	100, 241-242
वेद	four	104, 201, 206, 207, 229	100, 200-201, 207, 237
शक्र	fourteen	203	203
शर	five	191, 265	190, 279
शैल	seven	160, 199	149, 197-198
सागर	four	207	207
सूर्य	twelve	175, 199	165, 197-198
स्वर	seven	223	230

(Verse numbers and page numbers are as per [Līlā1937].)

Implications of Bhāskarācārya's *Līlāvatī* to the Common Core State Standards in Mathematics

Hari Prasad Koirala[*]

1 Introduction

After the publication of the Common Core State Standards for Mathematics ($CCSSM$, 2010) in the United States, over forty of the fifty states have adopted them in their school mathematics curriculum. The $CCSSM$ document is ambitious not only in terms of mathematical content in each grade level and development over time across grade levels, but also in terms of eight mathematical practices that are expected of every student irrespective of their grade level [CCS2010]. The purpose of this paper is to highlight how Bhāskarācārya's text, entitled the *Līlāvatī,* written in 1150 CE can be used to address some of the standards from the $CCSSM$ in the United States. The text contains topics from "arithmetic, elementary algebra, geometry, and mensuration" [Rao2004, p. 153]. It is written elegantly in poetry using everyday contexts such as flowers, forests, lakes, animals, people, sales, debts, and assets. All of these contexts have the potential to make mathematics more interesting and relevant to students even today. If students are convinced that mathematics is relevant to their lives, it can increase their perseverance in mathematical activities, which is one of the major goals of the $CCSSM$. This paper will discuss how some of the problems from the *Līlāvatī* have the potential to motivate students in today's world and engage them in meaningful mathematical activities.

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2 The CCSSM

The *CCSSM* mathematical contents from grades K–8 have been categorized under 11 mathematical domains: counting and cardinality, operations and algebraic thinking, number and operations in base 10, measurement and data, geometry, number and operations—fractions, the number system, expressions and equations, statistics and probability, ratios and proportional relationships, and functions. These topics are further explored at the high school level, which are divided into six conceptual categories: number and quantity, algebra, functions, modeling, geometry, and statistics and probability.

Among these topics, operations and algebraic thinking and geometry are given a high emphasis at the elementary school level. The number and operations in base 10 is expected in all grade levels K–5. The standards focus on whole numbers in grades K–2 and on fractions, decimals, measurement, ratio and proportions in grades 3–7. The *CCSSM* expects that the students will develop computational fluency using a variety of strategies [DP2012]. The *CCSSM* also argues that students need to reason mathematically and provide justifications of their thinking. It states: "one hallmark of mathematical understanding is the ability to justify, in a way appropriate to the student's mathematical maturity, why a particular mathematical statement is true or where a mathematical rule comes from" [CCS2010, p. 4]. If students are encouraged to explain and justify their mathematical thinking early on, their understanding of numbers and operations will be naturally extended to algebraic thinking in the middle and high schools.

Besides mathematical content appropriate to particular grade levels, the *CCSSM* expects that every student has the opportunity to be involved in making sense of problems and persevering in solving them using abstract and quantitative reasoning. The students are also expected to recognize and use mathematical structure in problem solving and communicating their ideas with others. They are also encouraged to use tools and models to enhance their mathematical understanding.

3 The *Līlāvatī*

Bhāskarācārya, born in India in 1114 CE and often referred to as Bhāskara II (hereafter Bhāskara), was one of the eminent mathematicians of the twelfth century. His most celebrated work, the *Līlāvatī*, has been very popular as an independent text because of its wide appeal and applications. This ancient text is composed of many computational rules and mathematical problems, rang-

ing from numbers and operations on whole numbers and fractions to quadratic equations and permutations and combinations. Several topics covered in the book are similar to the topics suggested by the *CCSSM* for elementary, middle, and high school mathematics. The major content domains suggested in the *CCSSM* but absent in the *Līlāvatī* are statistics and probability and functions.

Despite these similarities, the *Līlāvatī* uses a different approach of presentation compared to modern mathematics textbooks in at least three different ways. First, the *Līlāvatī* was not written for commercial purposes, it was written to teach mathematics to Bhāskara's daughter whose name was *Līlāvatī.* This was a genuine attempt of a father, a celebrated mathematician and astronomer, to motivate his daughter to learn mathematics and its importance in the world. She was considered an intelligent girl by Bhāskara and was addressed by him with love and affection throughout the book. He wished his daughter to be remembered by people for many generations in the future through this book. Second, unlike a modern mathematics textbook, the *Līlāvatī* is a book of mathematics written in poetry. The entire book consists of 275 verses [Col1993]. It integrates mathematics and poetry in an unusual way by showing the power of both poetry and mathematics to encourage creative thinking and problem solving. The book contributes enormously to the field of mathematics. However, it also contributes equally to the field of literature. Third, the book takes a balanced approach to expository teaching and problem solving. Throughout the book, simplicity, conciseness, and multiple strategies of solving problems are valued and encouraged. These ideas are illustrated in this paper by citing some Sanskrit verses from the *Līlāvatī.* The English translations of the chosen verses as well as their contemporary relevance within the context of the *CCSSM* are also provided. Discussed below are some mathematical principles valued by both the *Līlāvatī* and the *CCSSM*.

4 Deepening mathematical understanding through multiple approaches to computations

Although the *Līlāvatī* provides many direct rules without explanations, there are parts in the book that emphasize multiple strategies for computations in solving problems. In Chapter 2, the book provides eight operations of arithmetic, namely addition, subtraction, multiplication, division, square, square roots, cube, and cube roots. Although the rules and problems on addition and subtraction are brief, they emphasize the importance of adding numbers

based on the places of their digits. It is interesting that Bhāskara recommended adding and subtracting either from right to left or vice versa. Even though adding and subtracting from left to right is not a popular strategy used by textbooks and teachers in our society, it would be beneficial to use both strategies suggested by Bhāskara, given that many elementary school students and preservice teachers do not have an in-depth understanding of place value concepts [Tha2010]. Clearly, the emphasis here is not whether a particular operation is done from left to right or viceversa, but whether students understand the place value system and demonstrate computational fluency.

Perhaps the most notable implication of Bhāskara's *Līlāvatī* in elementary school mathematics is in the teaching of multiplication. The *CCSSM* intends that the third graders "understand properties of multiplication and the relationship between multiplication and division" (p. 23). In fourth grade, students are expected to "multiply a whole number of up to four digits by a one-digit whole number, and multiply two two-digit numbers, using strategies based on place value and the properties of operations" (p. 29). Just like in the *CCSSM*, Bhāskara emphasized multiple strategies and rules to perform multiplication, providing an opportunity for students to understand its nuances. In the *Līlāvatī* he described various methods of multiplication in two and a half stanzas. Considering the fact that he used only half a stanza to provide the rules of addition and subtraction, Bhāskara has given significant importance to multiplication in his book. He presented five different methods of multiplication, namely direct method, split method, factor method, place method, and the method of adding or subtracting [Līlā1989].

Provided below is a verse from the *Līlāvatī* (verse 16) to illustrate various methods of multiplication.

बाले बालकुरङ्गलोलनयने लीलावति प्रोच्यतां
पञ्चत्र्येकमिता दिवाकरगुणा अङ्काः कति स्युर्यदि।
रूपस्थानविभागखण्डगुणने कल्पासि कल्याणिनि
छिन्नास्तेन गुणेन ते च गुणिता अङ्काः कति स्युर्वद॥

bāle bālakuraṅgalolanayane līlāvati procyatāṃ
pañcatryekamitā divākaraguṇā aṅkāḥ kati syuryadi |
rūpasthānavibhāgakhaṇḍaguṇane kalpāsi kalyāṇini
chinnāstena guṇena te ca guṇitā aṅkāḥ kati syurvada ||

Beautiful and dear Līlāvatī, whose eyes are like a fawn's! Tell me numbers resulting from one hundred and thirty-five, taken into twelve, if thou be skilled in multiplication by whole or by parts, whether by subdivision of form or separation of digits. Tell me, auspicious woman, the quotient of the product divided by the same multiplier [Col1993, pp. 7–8].

The above problem can be solved by five different methods described by Bhāskara as follows:

1. Direct method: multiply 5 by 12 to get 60 ones, multiply 3 by 12 to get 36 tens and multiply 1 by 12 to get 12 hundreds. Add all of them to get 1620. This method also includes the *tatstha* method used by previous Hindu authors [Col1993].
2. Split method (subdivision of form): Split 12 into $8 + 4$ and multiply 135 by 8 to get 1080 and multiply 135 by 4 to get 540. Add these products to get 1620. This is the same as $135(8 + 4)$, the distributive property of multiplication over addition, which has been emphasized in the $CCSSM$. It is also important to note that 12 could have been split as $10 + 2$. That would be more efficient than splitting 12 as $8+4$. Bhāskara recommended splitting them into two convenient parts and given his emphasis on mathematical conciseness and the importance of the place value system it can be argued that he would have preferred to split 12 as $10 + 2$.
3. Factor method: 12 can be factored into 4 and 3. So first multiply 135 by 4 to get 540 and then multiply it by 3 to get 1620. This method simplifies the place values and makes the problem easier to solve.
4. Place method (taking the digits as parts): This is the same as the traditional algorithm that is commonly used in elementary classrooms. Multiply each digit of the multiplicand by each digit of the multiplier and add all of them carefully considering their place values.
5. Method of adding or subtracting: Add or subtract a convenient number to the multiplier to make the calculation easier and multiply the multiplicand by the new number. Then subtract or add the product of the added or subtracted number with the multiplicand. So 135×12 could be written in various forms, such as $135 \times 2 + 135 \times 10$ or $135 \times 20 - 135 \times 8$. This method is similar to the split method described above.

These methods also lead to $135\times 12 = 135\times 2(12\div 2)$; so multiply 270 by 6 and multiply 540 by 3. Students are expected to understand that the problem is mathematically identical if the multiplicand is multiplied by a number as long as the multiplier is divided by the same number. It is also noteworthy that Bhāskara suggested to reverse the multiplicand and the multiplier to make computations easier, basically applying the commutative rule of multiplication $(a \times b = b \times a)$.

After focusing on multiplication, he introduces division simply as an inverse process of multiplication, assuming that the students who are skillful in multiplication should have no difficulty understanding division. Besides this, he extends the factor method of multiplication to division. For example,

$1620 \div 12$ is the same as $405 \div 3$ (being reduced to least terms by the common measure 4). Through these examples, Bhāskara clearly indicated that varied methods of computations are important to enhance students' computational fluency and mathematical thinking. These methods are also emphasized by the *CCSSM*. Focusing on a variety of computational methods such as these enhances students' computational fluency and understanding, particularly in two *CCSSM* domains: Number & Operations in Base 10 and Operations and Algebraic Thinking intended for the elementary school.

5 Relevance of mathematics in day to day problems

The *Līlāvatī* is organized into 13 chapters. Like many mathematics textbooks, it begins with definitions of technical terms and measuring units such as money, weights, time, and distance. All of these terms were related to the day to day problems of that time. Even though the book was not intended for the masses, the problems presented there were easy to make sense of as they were practical and relevant to the individual learner and to the community at large. Bhāskara used various contexts such as people, sales, profit, travel, and religious information to pose his problems. Provided below is a problem (verse 52) based on the Hindu religion and culture that Bhāskara followed:

> अमलकमलराशेस्त्र्यंशपञ्चांशषष्ठैः
> त्रिनयनहरिसूर्या येन तुर्येण चार्या।
> गुरुपदमथ षड्भिः पूजितं शेषपद्मैः
> सकलकमलसंख्यां क्षिप्रमाख्याहि तस्य॥
>
> *amalakamalarāśestryaṃśapañcāṃśaṣaṣṭhaiḥ*
> *trinayanaharisūryā yena turyeṇa cāryā |*
> *gurupadamatha ṣaḍbhiḥ pūjitaṃ śeṣapadmaiḥ*
> *sakalakamalasaṃkhyāṃ kṣipramākhyāhi tasya ||*
>
> From a certain quantity of pure lotus flowers, a third were offered to Lord Shiva, a fifth to Lord Vishnu, a sixth to the sun, and a quarter to the goddess. The remaining six lotuses were offered to the guru. Tell me quickly the total number of lotuses.

Bhāskara provided the rule of supposition to solve these kinds of problems. Supposing that the whole quantity of lotus flowers is 1, the problem can be solved as follows:

$$1 - \left(\frac{1}{3} + \frac{1}{5} + \frac{1}{6} + \frac{1}{4}\right) = \frac{1}{20}.$$

Here, $\frac{1}{20}$th part of a whole is 6. Therefore the whole must be $6 \times 20 = 120$. His rule asked students to divide 6 by $\frac{1}{20}$, which yields 120.

This problem can be easily solved by using simple algebra as follows:

$$x - \left(\frac{x}{3} + \frac{x}{5} + \frac{x}{6} + \frac{x}{4}\right) = 6.$$

Solving for x, the total number of lotuses is 120. It is worth noting that these two solutions, numeric (by supposing the whole as 1) and algebraic (by supposing the whole as x) are very similar. In both cases students use supposition, 1 for the numeric solution and x for the algebraic solution. This maintains the structure of mathematics as suggested in the *CCSSM*. In grade 7, students are expected to "use variables to represent quantities in a real-world or mathematical problem, and construct simple equations and inequalities to solve problems by reasoning about the quantities" (p. 49). In grade 8, students are required to "solve linear equations with rational number coefficients, including equations whose solutions require expanding expressions using the distributive property and collecting like terms" (p. 54). If algebra is used, the problem above fits well with the 8th grade expectations from the *CCSSM*.

Even though the above problem is presented in a religious context to make it relevant to people, the context can be easily changed to fit the needs of students in the United States or elsewhere. It would be worthwhile to ask students to change the context of the problems based on their situation. The teacher may state, "This problem was created by Bhāskara in India in 1150 CE. How would you change the context of this problem so that the problem remains mathematically the same?" After this exercise students could be encouraged to develop new problems in which both the contexts and mathematics are new. Such activities encourage student's creativity and their ability to use language to formulate mathematical problems that are founded on everyday contexts of students.

Problems like these also provide an opportunity for students to connect arithmetic operations to algebra, which is one of the major goals of the *CCSSM*. Questions can be asked: In Bhāskara's method, can you suppose a number other than 1? Why or why not? Which method, numeric or algebraic, is easier to understand? Why? These questions will be helpful for students to understand that algebra is simply an extension of arithmetic and is more powerful in generalizing mathematics.

Provided below is another verse (verse 53), which is framed in the context of pilgrimage, taxes, and charity:

स्वार्धं प्रादात् प्रयागे नवलवयुगलं योऽवशेषाच्च काश्यां
शेषांघ्रिं शुल्कहेतोः पथि दशमलवान् षट् च शेषाद्गयायाम्।
शिष्टा निष्कत्रिषष्टिर्निजगृहमनया तीर्थपान्थः प्रयातः
तस्य द्रव्यप्रमाणं वद यदि भवता शेषजातिः श्रुताऽस्ति॥

svārdhaṃ prādāt prayāge navalavayugalaṃ yo'vaśeṣācca kāśyāṃ
śeṣāṃghriṃ śulkahetoḥ pathi daśamalavān ṣaṭ ca śeṣādgayāyām |
śiṣṭā niṣkatriṣaṣṭirnijagṛhamanayā tīrthapānthaḥ prayātaḥ
tasya dravyapramāṇaṃ vada yadi bhavatā śeṣajātiḥ śrutā'sti ||

A pilgrim gave half his money [to Brāhmins] at Prayāga. He spent two-ninths of the remainder at Kāśī and a quarter of the residue as taxes on the road. He then spent six-tenths of the remaining amount at Gayā. When he returned home from his pilgrimage, he was left with sixty-three *niṣkas*. If you have learned the method of fractional residues, state his original amount of money.

This problem is more complicated than the lotus flower problem presented above. Bhāskara suggested solving this problem by using the method of fractional residues as follows:

Suppose the pilgrim took 1 *niṣka* with him. Then,

$$\begin{aligned}
\text{Prayāga :} & 1-\frac{1}{2}=\frac{1}{2}\ \text{ left}\\
\text{Kāśī :} & \frac{2}{9}\times\frac{1}{2}=\frac{1}{9}; i.e.,\ \frac{1}{2}-\frac{1}{9}=\frac{7}{18}\ \text{ left}\\
\text{Taxes :} & \frac{1}{4}\times\frac{7}{18}=\frac{7}{72}; i.e.\ \frac{7}{18}-\frac{7}{72}=\frac{7}{24}\ \text{ left}\\
\text{Gayā :} & \frac{6}{10}\times\frac{7}{24}=\frac{7}{40}; i.e.\ \frac{7}{24}-\frac{7}{40}=\frac{7}{60}\ \text{ left.}
\end{aligned}$$

Therefore, the pilgrim had the amount of:

$$\frac{63}{\frac{7}{60}}=540\ \textit{niṣkas}.$$

Assuming that the pilgrim had only one *niṣka* to begin with could be confusing to students, as they would be thinking that the pilgrim must have taken more than one *niṣka* for his journey. To avoid this confusion, the teacher should ask some questions such as: Can this problem be solved by assuming other numbers, for example assuming that the pilgrim had 100 *niṣkas* to begin with? If we begin with 100 *niṣkas*, does it make the problem solving process easier or more difficult? What would be the rationale of assuming the total amount to be 1? The discussion of such questions will help students to understand that assuming one or other numbers is fine. However, assuming 100 as a whole could lead to more difficult computations because it is not divisible by 9 or 7. Depending on their background knowledge, this may motivate them to solve this problem algebraically by setting the original amount as x. The problem can be solved step by step as above or directly by:

$$x-\frac{x}{2}\rightarrow\left(\frac{x}{2}-\frac{2}{9}\times\frac{x}{2}\right)\rightarrow\left(\frac{7x}{18}-\frac{1}{4}\times\frac{7x}{18}\right)\rightarrow\left(\frac{21x}{72}-\frac{6}{10}\times\frac{21x}{72}\right)=63$$

$$\text{That means } \left(\frac{7x}{24} - \frac{7x}{40}\right) = 63 \to \frac{7x}{60} = 63$$
$$\text{or,} \qquad x = 540.$$

Even though this problem is more complicated than the lotus flower problem, this provides an opportunity for advanced mathematical content for students. It also allows the teacher to differentiate instruction based on a students' background knowledge of mathematics.

Making mathematics related to the real world was so important to Bhāskara that he provided many other problems based on everyday contexts that involved varieties of mathematical topics. He provided several rules and problems on purchase, sales, interest, and investment that were very practical. Here is one problem (verse 93) framed in the context of business investment that requires an understanding of ratio and proportion:

पञ्चाशदेकसहिता गणकाष्टषष्टिः
पञ्चोनिता नवतिरादिधनानि येषाम्।
प्राप्ता विमिश्रितधनैस्त्रिशती त्रिभिस्तैः
वाणिज्यतो वद विभज्य धनानि तेषाम्॥

pañcāśadekasahitā gaṇakāṣṭaṣaṣṭiḥ
pañconitā navatirādidhanāni yeṣām |
prāptā vimiśritadhanaistriśatī tribhistaiḥ
vāṇijyato vada vibhajya dhanāni teṣām ||

Three grocers invested 51, 68, 85 [*niṣkas*] respectively. Skillfully they increased their total assets to 300 [*niṣkas*]. State the shares of each.

Bhāskara gave the following rule (verse 92, half a stanza) to solve these kinds of problems:

प्रक्षेपका मिश्रहता विभक्ता प्रक्षेपयोगेन पृथक् फलानि॥

prakṣepakā miśrahatā vibhaktā prakṣepayogena pṛthak phalāni ||

An individual's share (after business) is the individual's investment multiplied by the total output and divided by the total investment.

Using this rule, the above problem can be solved as follows:

Total investment $= 51 + 68 + 85 = 204$. Total output $= 300$.

Therefore their shares are:

$$\frac{51 \times 300}{204} = 75; \ \frac{68 \times 300}{204} = 100; \ \frac{85 \times 300}{204} = 125.$$

This problem can also be solved algebraically by assuming a, b and c as individual investments and x as the total output. In this case, the solution would be:

$$\frac{ax}{a+b+c};\ \frac{bx}{a+b+c};\ \frac{cx}{a+b+c}.$$

Even though Bhāskara wrote his book more than 800 years ago, these problems are still relevant in today's society. Over this period, several standards for teaching mathematics have come and gone. However, no standards in school mathematics have undermined the relevance of mathematics in the everyday world.

6 Teaching mathematics by drawing examples from nature

Bhāskara seemed to have great love for nature and natural phenomena, which made him compose several problems involving nature, bees, butterflies, birds, and other animals. An example (verse 68) is provided below:

> अलिकुलदलमूलं मालतीं यातमष्टौ
> निखिलनवमभागाश्चालिनी भृङ्गमेकम्।
> निशि परिमललुब्धं पद्ममध्ये निरुद्धं
> प्रतिरणति रणन्तं ब्रूहि कान्तेऽलिसंख्याम्॥
>
> *alikuladalamūlaṃ mālatīṃ yātamaṣṭau*
> *nikhilanavamabhāgāścālinī bhṛṅgamekam* |
> *niśi parimalalubdhaṃ padmamadhye niruddhaṃ*
> *pratiraṇati raṇantaṃ brūhi kānte'lisaṃkhyāṃ* ||
>
> From a swarm of black bees, the square root of half the total went to a shrub of jasmine. So did eight-ninths of the whole swarm. A male bee captivated by the fragrance was confined within a lotus because it closes at night. The scared black bee began to hum. A female bee heard the humming and started to buzz in response. O beloved, how many bees were there?

This problem is fairly complex. Bhāskara provided the rule for assimilation of the root's coefficients in verses 62–63, which essentially requires a solution to the problem by completing the squares on both sides of the equation. The rule specifically states "to add the square of half the multiplier of the root to the given number"[Col1993, p. 38]:

Let the number of bees in the swarm be x. Then,

$$x-\sqrt{\frac{1}{2}x}-\frac{8}{9}x=2 \rightarrow \frac{x}{9}-\sqrt{\frac{1}{2}x}=2.$$

Setting $y=\frac{1}{2}x$, we get,

$$\frac{2y}{9} - \sqrt{y} = 2 \rightarrow y - \frac{9}{2}\sqrt{y} = 9.$$

Because this is a standard form, we can use Bhāskara's rule and add $\left(\frac{9}{4}\right)^2$ on both sides.

$$y - \frac{9}{2}\sqrt{y} + \left(\frac{9}{4}\right)^2 = 9 + \left(\frac{9}{4}\right)^2$$
$$\left(\sqrt{y} - \frac{9}{4}\right)^2 = \left(\frac{15}{4}\right)^2.$$

Hence, $y = 36$ or $x = 72$. Thus the number of bees in the swarm is 72.

These kinds of problems and solutions fit well with high school algebra in the *CCSSM*, particularly the domain of "Reasoning with Equations and Inequalities" abbreviated as $A-REI$. The following two standards are directly from the *CCSSM* [CCS2010, p. 65]:

> Solve quadratic equations in one variable.
>
> a. Use the method of completing the square to transform any quadratic equation in x into an equation of the form $(x-p)^2 = q$ that has the same solutions. Derive the quadratic formula from this form.
>
> b. Solve quadratic equations by inspection (e.g., for $x^2 = 49$), taking square roots, completing the square, the quadratic formula and factoring, as appropriate to the initial form of the equation.

While this problem fits well in the common core state standards, the problem posed by Bhāskara has many interesting aspects. First the problem is presented in a verse using natural phenomena. It almost reads like a story and presents an account of an unfortunate situation two bees are going through. It shows the love the bees offer to each other and gives life to a mathematical problem that would otherwise be dull and routine.

Bhāskara noted that these kinds of problems are easier to solve, if solved for x^2 rather than for x. An opportunity for further mathematical thinking can be created if the teacher asks students: Is it better to solve this problem for x^2 or x? Why? After some discussion students can realize that solving this problem for $2x^2$ saves some steps.

Let the number of bees in the swarm be $2x^2$. Then,

$$2x^2 - x - \frac{8}{9}\left(2x^2\right) = 2 \rightarrow \frac{2x^2}{9} - x = 2 \rightarrow x^2 - \frac{9}{2}x = 9.$$

Because this is a standard form, we can use Bhāskara's rule and add $\left(\frac{9}{4}\right)^2$ on both sides.

$$x^2 - \frac{9}{2}x + \left(\frac{9}{4}\right)^2 = 9 + \left(\frac{9}{4}\right)^2$$
$$\left(x - \frac{9}{4}\right)^2 = \left(\frac{15}{4}\right)^2.$$

Hence, $x = 6$ or $2x^2 = 72$. Therefore the number of bees in the swarm is 72.

This problem not only fits well with the algebra standards from the *CCSSM* but also can create a good opportunity for differentiating instruction. If students are struggling to solve this problem algebraically, the solution can be obtained by guessing and checking. Because the total number of bees in the swarm was the square root of half the total, the logical guesses would be the numbers that are two times as large as the perfect squares. The first 10 perfect squares are 1, 4, 9, 16, 25, 36, 49, 64, 81, and 100. Therefore the possible numbers of bees in the swarm are 2, 8, 18, 32, 50, 72, 98, 128, 162, and 200. The problem also states that eight-ninths of the total went to a shrub of jasmine, which implies that the total number of bees must be divisible by 9. This leaves only three possible solutions from the above set of numbers: 18, 72, and 162. Out of these three numbers, only 72 satisfies the equation: $\frac{x}{9} - \sqrt{\frac{1}{2}x} = 2$. Therefore the total number of bees must be 72. In case the solution was not obtained by using the first 10 perfect squares, another set of perfect squares could have been tried. It appears that Bhāskara chose the numbers to be small enough so that the solution could also be obtained by guessing and checking.

This problem is but one example from the *Līlāvatī* that integrates natural phenomena with mathematical problem solving. There are many other context-based problems in the text which are consistent with the goals of the *CCSSM* in that the purpose of school mathematics is to help students to recognize and utilize the power of mathematical reasoning.

7 Problem integrating different areas of mathematics

As described above, Bhāskara presented many problems that naturally extended from arithmetic to algebra. He clearly distinguished between *vyaktagaṇita* (arithmetic) and *avyaktagaṇita* (algebra), and wrote a separate book on *Bījagaṇita* (algebra) focusing on the study of unknown quantities. Bhāskara

remarked in his *Bījagaṇita* that "mathematicians have declared algebra to be calculation accompanied by proofs; otherwise, there would be no distinction between arithmetic and algebra" [BīPa2012, 8]. Even though Bhāskara distinguished between arithmetic and algebra by writing two books, one in arithmetic and one in algebra, some basic rules and concepts provided in the *Līlāvatī* are repeated again in the *Bījagaṇita*. It is clear that Bhāskara's work shows how algebra grows out of arithmetic. The *CCSSM* also clearly indicates this relationship between arithmetic and algebra. The students are expected to understand the structure of numbers and develop computational fluency in the elementary grade levels and develop a more comprehensive understanding of those structures using algebraic principles in the middle and high school.

Although it is important to integrate arithmetic and algebra as discussed above, mathematics does not end by integrating only these two areas. Challenging mathematical problems integrate arithmetic, algebra, and geometry in a way people can understand. Bhāskara brought these pieces together in several problems including the one (verse 150) presented below:

> अस्ति स्तंभतले बिलं तदुपरि क्रीडाशिखण्डी स्थितः
> स्तंभे हस्तनवोच्छ्रिते त्रिगुणितस्तंभप्रमाणान्तरे ।
> दृष्ट्वाहिं बिलमाव्रजन्तमपतत् तिर्यक् स तस्योपरि
> क्षिप्रं ब्रूहि तयोर्बिलात् कतिमितैः साम्येन गत्योर्युतिः ॥
>
> *asti stambhatale bilaṃ tadupari krīḍāśikhaṇḍī sthitaḥ*
> *stambhe hastanavocchrite triguṇitastambhapramāṇāntare |*
> *dṛṣṭvāhiṃ bilamāvrajantamapatat tiryak sa tasyopari*
> *kṣipraṃ brūhi tayorbilāt katimitaiḥ sāmyena gatyoryutiḥ ||*
>
> There was a snake hole at the foot of a pole, nine cubits high. A domesticated peacock was sitting on its summit. Seeing a snake at the distance of thrice the pole crawling towards the hole, the peacock obliquely pounced on the snake. If the peacock and the snake moved an equal distance, say quickly, how far from the hole did they meet?

This problem can be presented as shown in the diagram in Figure 1. The problem integrates arithmetic, algebra, and geometry very well. At the same time, it is set in an appealing context, which makes the application of the Pythagorean theorem much more interesting.

The problem can be solved using the theorem as follows:

$$\begin{aligned}(27-x)^2 &= x^2+9^2\\ (27)^2-54x+x^2 &= x^2+81\\ 27(27-2x-3) &= 0\\ \text{(or)}\quad x &= 12.\end{aligned}$$

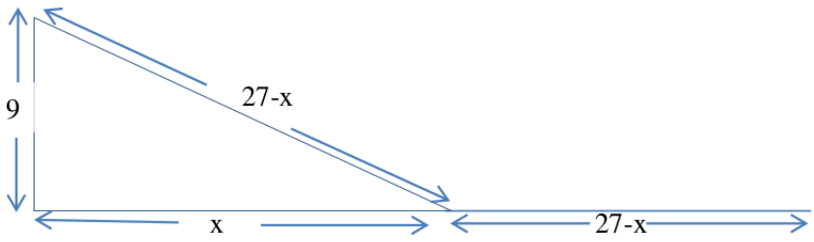

Figure 1: The peacock and snake problem.

So the peacock pounced on the snake at 12 cubits away from the hole i.e., the base of the pole. Besides integrating arithmetic, algebra, and geometry, this problem also introduces the concept of mathematical modeling. We know for sure that the actual path traced by the peacock cannot be a straight line and also the fact that the peacock and the snake cannot move at the same speed. However, by formulating such creative mathematical problems the students come to understand that certain simplifying assumptions need to be made. Such a situation creates an interesting scenario for mathematical modeling which is one of the six content standards recommended by the *CCSSM* for high school mathematics. The *CCSSM* states that "mathematically proficient students who can apply what they know are comfortable making assumptions and approximations to simplify a complicated situation" (p. 7). Although Bhāskara did not use the term mathematical modeling in his writing, the solutions to some of his problems require it.

This problem also allows students to realize that mathematical structures are valuable in all branches of mathematics, which is an important goal envisioned by the *CCSSM*. It is also important to note that the *Līlāvatī* was written in verses to address questions in science (astronomy) that required mathematics. The verses in the *Līlāvatī* perfectly follow the meter and prosody of poetic traditions in India during that time. When the verses are read with appropriate rhythms, styles, intonations, and facial expressions, they add a new dimension to mathematical problems. When the verses are read following these traditional standards, people consider the reading as a form of performing arts. To make it more interesting, music can be added. So the mathematics becomes a playful activity. The *Līlāvatī* is an exceptional piece of work that integrates several branches of mathematics with science (astronomy), literature (poems), and performing arts. If students have the opportunity to learn mathematics while integrating these various areas, there is a good chance that they will be motivated to learn mathematics and persevere in solving problems as emphasized in the *CCSSM*.

8 Integration of intuitive and formal approaches to mathematics

Unlike geometry in ancient Greece, the *Līlāvatī* does not use an axiomatic approach to mathematics. The rules and problems are presented intuitively. Even though diagrams were not used, the descriptions of problems through verses were highly visual. Because of Bhāskara's poetic ability, the situation and contexts of the mathematical problems are descriptive and allow the learner to construct visual imagery of the problem situation. For example, the swarm of bees problem (verse 68) and the peacock and snake problem (verse 150) are so descriptive that it is not difficult for students to construct visual imagery of the problems in their mind. Construction of visual imagery of the problem situation helps students to make sense of mathematical problems which is one of the mathematical practices suggested by the *CCSSM*.

As stated earlier, the problems in the *Līlāvatī* use an intuitive approach to mathematics. For example, in the fraction problems (verses 52 and 53), Bhāskara suggested solving the problems by assuming the original quantity to be 1, which represents a whole. Even though it is possible to solve these problems by assuming other numbers, the process is simplest if the total is assumed to be 1. Nevertheless, as discussed above, these problems can be solved by using algebra, which is a more formal approach to mathematics. Problem after problem, the *Līlāvatī* provides an opportunity for students to integrate intuitive and formal approaches to mathematics. This promotes the idea that mathematics is a sense making process, and at the same time it is also a discipline that values abstract reasoning. Both of which are hallmarks of the *CCSSM*.

9 Conclusions

The purpose of this paper was to show how the problems from Bhāskara's *Līlāvatī,* written more than 850 years ago have the potential to help students achieve some of the content domains and mathematical practices as outlined in the *CCSSM*. As shown throughout this paper, the problems in the *Līlāvatī* can be solved by applying the rules provided by Bhāskara and also by applying modern algebraic principles. If students are given an opportunity to solve these problems both by using Bhāskara's rules and algebraic procedures, they will not only realize how the same problem can be solved in multiple ways, but also understand the inherent relationship between numeric operations and algebraic thinking, which is one of the goals outlined by the *CCSSM*.

Even though the *Līlāvatī* did not necessarily explain why the rules worked, most of these rules can be easily connected to modern algebraic principles as discussed throughout this paper. What is really unusual about the *Līlāvatī* is that it provides mathematical rules and problems using beautiful verses that combine mathematics with nature, people, business, and art. Such a combination of mathematics with our everyday world can be fascinating to students around the world and help them persevere in solving problems as envisioned by the *CCSSM*.

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